Computational Electromagnetics

Center for Integrated Access Networks

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Outline

- Introduction
- Maxwell's equations
- Finite Difference Time Domain method
- Integral equation methods
- Finite element method

- **Books:**
  - J. M. Jin, Theory and computation of electromagnetic fields, Wiley, IEEE Press, 2010.
  - Computational Methods in Electromagnetics, A. F. Peterson, S. Ray and R. Mittra, IEEE Press, Wiley
  - Finite Element Methods in Electromagnetics, J. Jin, Wiley, Second Edition
  - Computational Electrodynamics: Finite Difference Time Domain Method, A. Taflov, Artech House
- **Acknowledgements:**
  - Prof. J. M. Jin for providing an access to his lecture notes. These notes were used in various parts of this short course.

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- Finite element method

Introduction

- Wireless communication
- High-speed high-density circuits
- Defense
- Remote sensing
- Medical applications
- Photonics

Introduction: Stealth technology

F117 Night Hawk

B-2 Spirit

F-22 Raptor

Joint Strike Fighter

Introduction: MRI System

Introduction

- **Photonics:**

  Waveguides and fibers

  Resonators

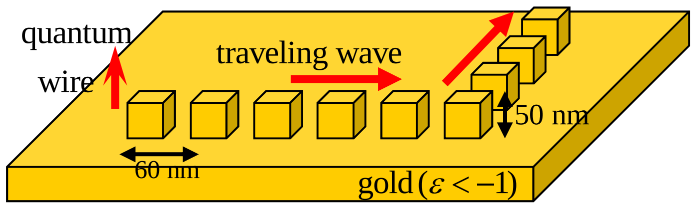
  Lasers

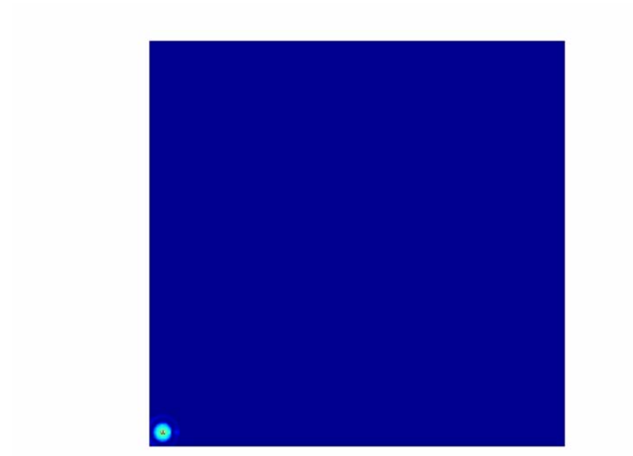
  Filters

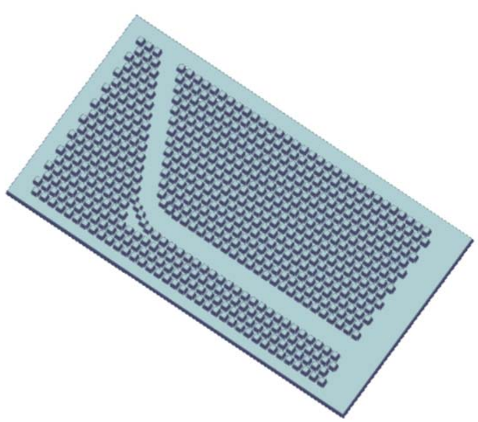
  Couplers

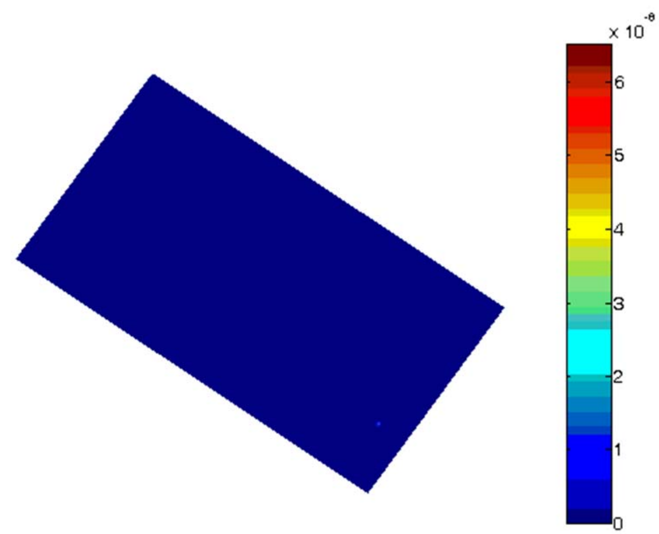
  Etc, etc, etc

Introduction: Photonic waveguides









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Maxwell's Equations

Maxwell's equations in differential form:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} - \vec{M}$$ **Faraday's law**

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$ **Maxwell-Ampere's law**

$$\nabla \cdot \vec{D} = \rho$$ **Gauss's law**

$$\nabla \cdot \vec{B} = \rho_m$$ **Gauss's law-magnetic**

Constitutive relations:

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H} \qquad \vec{J}_c = \sigma \vec{E}$$

Boundary Conditions

$$\hat{n}\times(\vec{E}_2-\vec{E}_1)=0 \qquad (\text{or} \quad -\vec{M}_s)$$
$$\hat{n}\times(\vec{H}_2-\vec{H}_1)=\vec{J}_s$$
$$\hat{n}\cdot(\vec{D}_2-\vec{D}_1)=\rho_s$$
$$\hat{n}\cdot(\vec{B}_2-\vec{B}_1)=0 \qquad (\text{or} \quad \rho_{ms})$$

Time-Harmonic Fields (1)

Euler's formula: $e^{j\alpha} = \cos\alpha + j\sin\alpha$

$$\cos\alpha = \mathrm{Re}(e^{j\alpha})$$

If $V(x,y,z,t)$ is oscillating at a single frequency:

$$\begin{aligned} V(x,y,z,t) &= V'(x,y,z)\cos(\omega t+\alpha) \\ &= V'(x,y,z)\,\mathrm{Re}\left(e^{j(\omega t+\alpha)}\right) \\ &= \mathrm{Re}\left[V'(x,y,z)e^{j\alpha}e^{j\omega t}\right] \\ &= \mathrm{Re}\left[\hat{V}(x,y,z)\,e^{j\omega t}\right] \end{aligned}$$

$\hat{V}(x,y,z)$: phasor or complex quantity

Time-Harmonic Fields (2)

Extension to vectors:

$$E_{\substack{x\\y\\z}}(x,y,z,t) = \mathrm{Re}[\hat{E}_{\substack{x\\y\\z}}(x,y,z)e^{j\omega t}]$$

$$\vec{E}(x,y,z,t) = \mathrm{Re}[\hat{\vec{E}}(x,y,z)e^{j\omega t}]$$

$$\vec{H}(x,y,z,t) = \mathrm{Re}[\hat{\vec{H}}(x,y,z)e^{j\omega t}]$$

} Time-harmonic field

Same expressions for other quantities, such as **D, B, J, M**

Time-Harmonic Fields (3)

$$\nabla \times \hat{\vec{E}} = -j\omega\hat{\vec{B}} - \hat{\vec{M}}$$

$$\nabla \times \hat{\vec{H}} = j\omega\hat{\vec{D}} + \hat{\vec{J}}$$

$$\nabla \cdot \hat{\vec{J}} = -j\omega\hat{\rho}_e$$

$$\nabla \cdot \hat{\vec{M}} = -j\omega\hat{\rho}_m$$

Simplify notation

$$\nabla \times \vec{E} = -j\omega\vec{B} - \vec{M}$$

$$\nabla \times \vec{H} = j\omega\vec{D} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho_e$$

$$\nabla \cdot \vec{B} = \rho_m$$

$$\nabla \cdot \vec{J} = -j\omega\rho_e$$

$$\nabla \cdot \vec{M} = -j\omega\rho_m$$

Maxwell's equations for time-harmonic fields

Auxiliary Potential Functions (1)

Given: $\vec{J}$, $\vec{M}$

Solve: $\nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J}$

$$\nabla^2 \vec{F} + k^2 \vec{F} = -\varepsilon \vec{M}$$

Calculate:

$$\vec{E} = \vec{E}_e + \vec{E}_m = -j\omega \vec{A} - \frac{j}{\omega\mu\varepsilon} \nabla(\underbrace{\nabla \cdot \vec{A}}_{-j\omega\rho}) - \frac{1}{\varepsilon} \nabla \times \vec{F}$$

$$\vec{H} = \vec{H}_e + \vec{H}_m = \frac{1}{\mu} \nabla \times \vec{A} - j\omega \vec{F} - \frac{j}{\omega\mu\varepsilon} \nabla(\underbrace{\nabla \cdot \vec{F}}_{-j\omega\rho_m})$$

Current-charge relation

$$\nabla \cdot \vec{A} = -j\omega\rho$$

$$\nabla \cdot \vec{F} = -j\omega\rho_m$$

Solution of Vector Potential (2)

$$\vec{A}(\vec{r}) = \frac{\mu}{4\pi} \iiint_V \vec{J}(\vec{r}\,') \frac{e^{-jkR}}{R} dv'$$

$$\vec{F}(\vec{r}) = \frac{\varepsilon}{4\pi} \iiint_V \vec{M}(\vec{r}\,') \frac{e^{-jkR}}{R} dv'$$

$$\vec{E} = -j\omega\vec{A} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{A}) - \frac{1}{\varepsilon}\nabla\times\vec{F}$$

$$\vec{H} = \frac{1}{\mu}\nabla\times\vec{A} - j\omega\vec{F} - \frac{j}{\omega\mu\varepsilon}\nabla(\nabla\cdot\vec{F})$$

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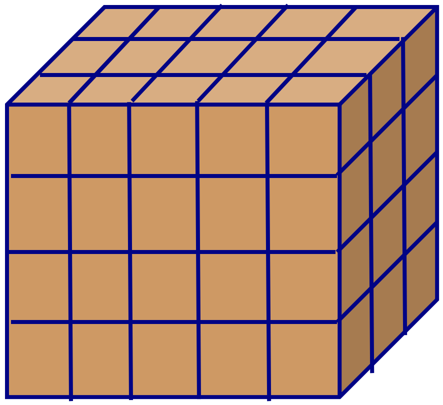
- **What is FDTD?**
  - **Direct solution of Maxwell's equations in the time domain**
  - **Approximate the time and spatial derivatives by finite differences**
  - **Truncate the computational domain and simulate the infinite space by Absorbing Boundary Conditions (ABCs)**

- **Advantages**
  - **Can be applied to general structures and materials**
  - **Can handle non-linear problems and structures with time varying parameters**
  - **Provides broadband information**
  - **Requires low memory → can be used for very large problems (e.g. billions of unknowns can be handled)**
  - **Relatively easy to implement**
  - **Computational cost (~$O(N_tN)$)**

- **Challenges**
  - **In conventional FDTD the geometry must confirm to the grid → staircase approximation**
  - **Can be applied to general structures and materials**
  - **The grid needs to be truncated to model outgoing conditions (due the differential equation formulation)**
  - **Basic algorithm leads to numerical dispersion and anisotropy**
  - **Time step is constrained by the spatial discretization, dt<c dt/sqrt(3) → many time steps for subwavelength domains**

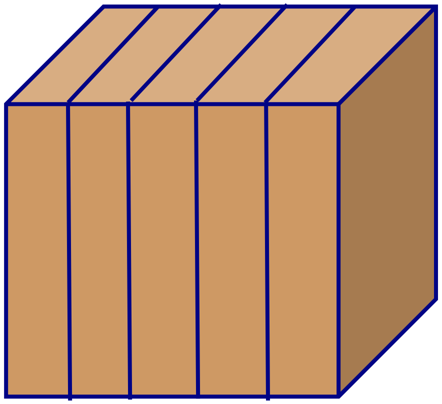
- **Approach**
  - **Divide the computational domain into bricks**
  - **Assign fields to every grid node and time step**
  - **Discretize continuous time domain Maxwell's equations by using finite differences**
  - **Apply ABC for the exterior surfaces**
  - **Solve within the computational domain for every time step at every grid location**



- **One dimensional (1D) case:**
  - **Consider configuration that is infinite in y and z but finite in x, e.g. a stack of layers**
  - **Assume no variations in y and z, i.e.** $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$
  - **Assume a plane wave with the electric field in z, magnetic field in y, and the propagation in x**
  - **Maxwell's equations are reduced to**

$$\frac{\partial H}{\partial t} = \frac{1}{\mu}\frac{\partial E}{\partial x}$$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H}{\partial x}$$



  - **The wave equation is reduced to**

$$\frac{\partial^2 H}{\partial t^2} = \frac{1}{\varepsilon\mu}\frac{\partial^2 H}{\partial x^2} \Rightarrow \frac{\partial^2 H}{\partial t^2} = c^2\frac{\partial^2 H}{\partial x^2}$$

- **Numerical procedure**
  - **Maxwell's equations are continuous equations**
  - **Approximate continuous derivatives by finite differences**
  - **Consider** $\frac{\partial H}{\partial t} = \lim_{\Delta t \to 0} \frac{\Delta H}{\Delta t}$
  - **Backward difference: first order accurate**

$$\frac{\partial H}{\partial t} \approx \frac{H(t) - H(t - \Delta t)}{\Delta t} + O(\Delta t)$$

  - **Forward difference: first order accurate**

$$\frac{\partial H}{\partial t} \approx \frac{H(t + \Delta t) - H(t)}{\Delta t} + O(\Delta t)$$

  - **Central difference: second order accurate!**

$$\frac{\partial H}{\partial t} \approx \frac{H(t + \Delta t) - H(t - \Delta t)}{2\Delta t} + O(\Delta t^2);$$

$$\frac{\partial^2 H}{\partial t^2} \approx \frac{H(t + \Delta t) - 2H(t) + H(t - \Delta t)}{\Delta t^2} + O(\Delta t^2)$$

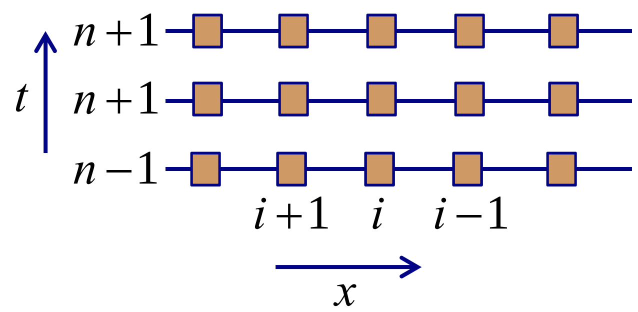
- # Derivation
  - Use Taylor expansion for forward and backward fields

$$H(t+\Delta t) \approx H(t) + \Delta t H'(t) + \frac{1}{2}(\Delta t)^2 H''(t) + \frac{1}{6}(\Delta t)^3 H'''(t)$$

$$H(t-\Delta t) \approx H(t) - \Delta t H'(t) + \frac{1}{2}(\Delta t)^2 H''(t) - \frac{1}{6}(\Delta t)^3 H'''(t)$$

  - Subtract $\frac{H(t+\Delta t)-H(t)}{\Delta t}$ to obtain forward difference
  - Subtract $\frac{H(t)-H(t-\Delta t)}{\Delta t}$ to obtain backward difference
  - Subtract $\frac{H(t+\Delta t)-H(t-\Delta t)}{2\Delta t}$ and $\frac{H(t+\Delta t)-2H(t)+H(t-\Delta t)}{\Delta t^2}$ to obtain central differences

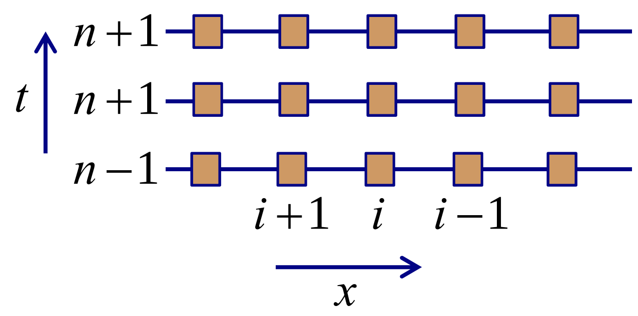
- # FDTD for 1D wave equation
  - **Equation:** $\frac{\partial^2 H}{\partial t^2} = c^2 \frac{\partial^2 H}{\partial x^2}$
  - **Notations:**
    - $t = n\Delta t, x = i\Delta x$
    - $H(n\Delta t, i\Delta x) = H_i^n$
    - $E(n\Delta t, i\Delta x) = E_i^n$
  - **discretization**



$$\frac{H_i^{n+1} - 2H_i^n + H_i^{n-1}}{(\Delta t)^2} = c^2 \frac{H_{i+1}^n - 2H_i^n + H_{i-1}^n}{(\Delta x)^2}$$

$$\Rightarrow H_i^{n+1} = \underbrace{\left(\frac{c\Delta t}{\Delta x}\right)^2}_{\gamma^2} \left(H_{i+1}^n + H_{i-1}^n\right) + 2\left(1 - \gamma^2\right) H_i^n - H_i^{n-1}$$

- **FDTD for 1D wave equation (cont'd)**
  - **Properties**
    - **Fully explicit second-order accurate scheme**
    - **Two previous time steps are required to be stored**
    - **The analysis is recursive in time**
    - **Each field is found at every time step at every node**

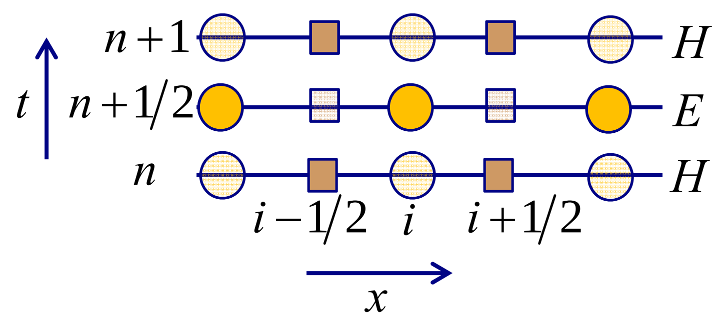


FDTD for 1D Maxwell's equations

- **Equation:** $\frac{\partial H}{\partial t} = \frac{1}{\mu}\frac{\partial E}{\partial x};\qquad \frac{\partial E}{\partial t} = \frac{1}{\varepsilon}\frac{\partial H}{\partial x}$
- **Notations:**
  - Redefine: $2\Delta t \rightarrow \Delta t$
  - Central difference: $\frac{\partial H}{\partial t} \approx \frac{H(t+\Delta t/2) - H(t-\Delta t/2)}{\Delta t}$
  - Involved time steps: $t = n\Delta t, t = n\Delta t + \Delta t/2$
  - Involved spatial steps: $x = i\Delta x, x = i\Delta x + \Delta x/2$
  - Involved fields:

$$H(n\Delta t, i\Delta x + \Delta x/2) = H^{n}_{i+1/2}$$

$$E(n\Delta t + \Delta t/2, i\Delta x) = E^{n+1/2}_{i}$$

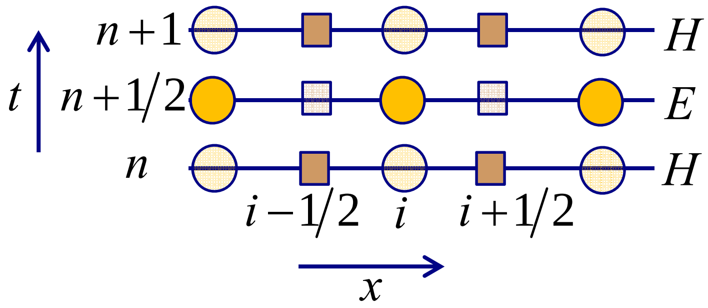


FDTD for 1D Maxwell's equations (cont'd)

- **Update equations:**

$$H_{i+1/2}^{n+1} = H_{i+1/2}^{n} + \frac{\Delta t}{\Delta x}\frac{1}{\mu}\left(E_{i+1}^{n+1/2} - E_{i}^{n+1/2}\right)$$

$$E_{i}^{n+1/2} = E_{i}^{n-1/2} + \frac{\Delta t}{\Delta x}\frac{1}{\varepsilon}\left(H_{i+1/2}^{n} - H_{i-1/2}^{n}\right)$$



- **Yee algorithm:**
  - The fields are given in different space and time locations
  - Interleaved E & H grids
  - Leapfrog in time
  - Explicit stepping → no matrix inversions, no iterations
  - The solution is more robust and straightforward
  - E & H boundary conditions are enforced automatically

Numerical dispersion of FDTD

- Consider a time harmonic wave $H = e^{j(\omega t - kx)}$
- Continuous dispersion relation is linear: $\omega = ck$
- Numerical wave: $H_i^n = e^{j(\omega n\Delta t - ki\Delta x)}$
- Use the FDTD update equation $H_i^{n+1} = \gamma^2\left(H_{i+1}^n + H_{i-1}^n\right) + 2\left(1-\gamma^2\right) - H_i^{n-1}$
- → $e^{j\omega\Delta t} = \gamma^2\left(e^{-jk\Delta x} + e^{jk\Delta x}\right) + 2\left(1-\gamma^2\right) - e^{-j\omega\Delta t}$
- → $\boxed{\sin^2\left(\frac{\omega\Delta t}{2}\right) = \gamma^2 \sin^2\left(\frac{k\Delta x}{2}\right)}$
- Properties:
  - In general the phase velocity depends on the frequency → numerical dispersion!
  - $\Delta t, \Delta x \to 0$ → $\omega^2 = (ck)^2$- weak dispersion
  - $\Delta x = c\Delta t, \gamma = 1$→ "magic" time step with no dispersion (for the 1D case)

- **Stability of FDTD**
  - **Consider** $H = e^{j(\omega t - kx)}$ **and** $H_i^n = e^{j(\omega n\Delta t - ki\Delta x)}$
  - **Allow** ω **to be complex, i.e.** $\omega = \omega' + j\omega''$
  - **The propagation is stable if** $\omega'' \geq 0$ **(wave**$\sim e^{-|\omega'' t|}$**)**
  - **For** $\omega'' < 0$ **(wave**$\sim e^{|\omega'' t|}$**) the propagation is unstable**
  - **Define** $\Omega = \omega\Delta t/2, K = k\Delta x/2, g = e^{-j\Omega}$ **and use** $\sin\Omega - \gamma\sin K = 0$
  - **→** $g^2 + 2j\gamma g\sin K - 1 = 0 \Rightarrow g = -jK\sin K \pm\sqrt{1-\gamma^2\sin^2 K}$
  - **Stable solution:** $|g| \leq 1 \Rightarrow 1-\gamma^2\sin^2 K \geq 0 \Rightarrow \gamma \leq 1$
  - **Courant stability criterion for 1D case:**

$$\boxed{\Delta t \leq \frac{\Delta x}{c}}$$

- **Stability of FDTD (cont'd)**
  - **Physical meaning: the wave should not pass through more than one cell in a single time step**
  - **Under the magic time step the scheme is exact**
- **How to choose Δx and Δt**
  - **If no fine geometrical features are present:**

    $\Delta x < \lambda_{\min}/10$ **to satisfy Nyquist sampling criterion**
  - **If fine geometrical features are present:**

    Δx **is determined by the features**
  - **Choose $\Delta t < \Delta x/c$ (note that for fine geometrical features, Δx is small and the time step is small as well!)**

- **Source modeling**
  - **Initial conditions: Define E_i^0, H_i^1 for all i.**
  - **Hard source**
    - **Specify fields at certain locations independent of surrounding cells, i.e. replace the Yee equations, e.g. $E_{z,i_s}^n = F(t_n)$**
    - **Easy to implement but lead to reflections**
  - **Soft source**
    - **Sources that allow field updates, e.g. current sources**
    - **Equations are updated and no reflections but harder to implement**
  - **Plane wave injection: Scattered field / total field approach**

- **Time dependence of sources**
  - **Time harmonic sources**
    - **Simple sin functions leads to discontinuities and a relatively broadband spectrum**
    - **Use sin with a smooth transition function ($g(t)$)**

$$F(t) = F_0 \sin(\omega_0 n \Delta t) g(t)$$

  - **Pulsed sources**
    - **Gaussian: Contains DC → a problem**
    - **Time derivative of a Gaussian → no DC**
    - **Modulated Gaussian**

- **Domain truncation: Perfectly matched layer (PML)**
  - **Add an additional domain where a truncation is needed**
  - **Fill the domain with a matching lossy material with σ & σ_e satisfying** $\frac{\sigma_m}{\mu_1} = \frac{\sigma}{\varepsilon_1}$ & $\sqrt{\frac{\mu_1}{\varepsilon_1}} = \sqrt{\frac{\mu_2}{\varepsilon_2}}$

  - **→ No reflections are obtained because**

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}\frac{1+\sigma_m/(j\omega\mu_2)}{1+\sigma/(j\omega\varepsilon_2)}}$$

  - **This idea can be extended to 2D and 3D but the lossy materials should be anisotropic to match all directions**
  - **Other absorbing boundary conditions can be used as well**

3-D FDTD Algorithm

Maxwell's equations:

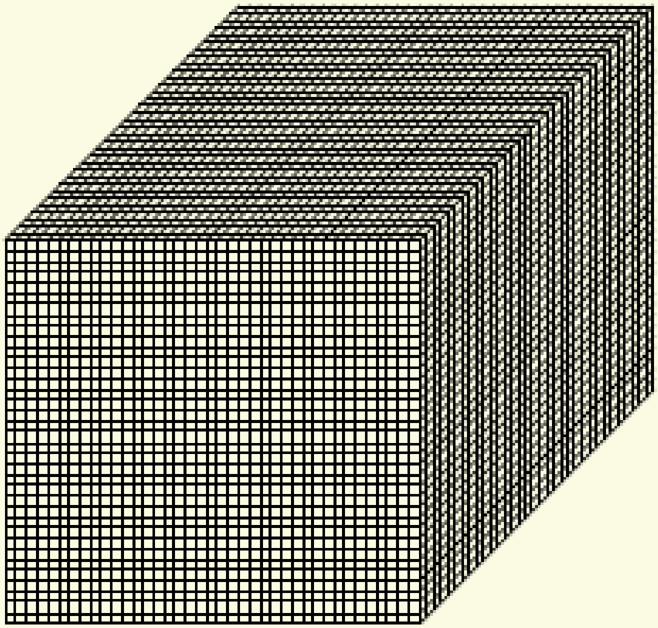
$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial \mathcal{H}_x}{\partial t} \qquad \frac{\partial \mathcal{H}_z}{\partial y} - \frac{\partial \mathcal{H}_y}{\partial z} = \varepsilon \frac{\partial \mathcal{E}_x}{\partial t} + \sigma \mathcal{E}_x + \mathcal{J}_x$$

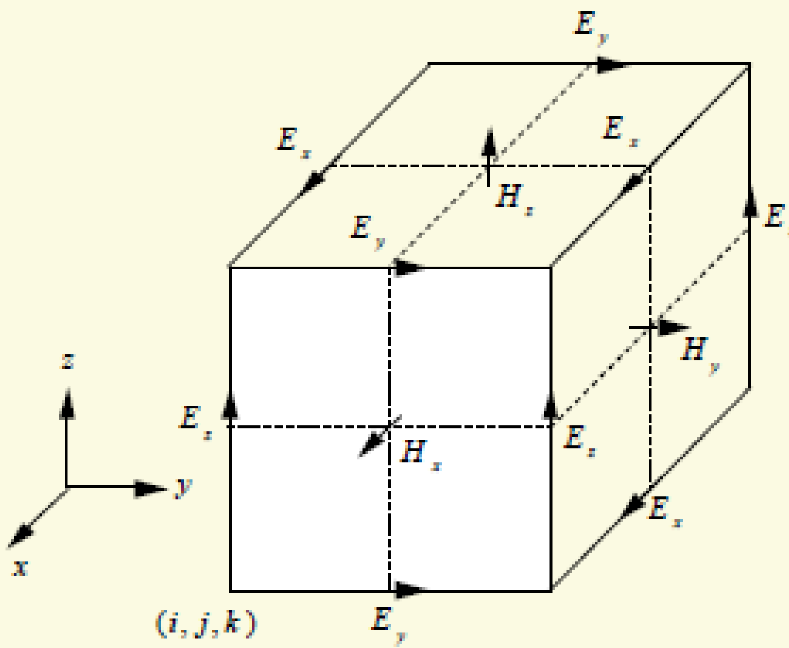
$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\mu \frac{\partial \mathcal{H}_y}{\partial t} \qquad \frac{\partial \mathcal{H}_x}{\partial z} - \frac{\partial \mathcal{H}_z}{\partial x} = \varepsilon \frac{\partial \mathcal{E}_y}{\partial t} + \sigma \mathcal{E}_y + \mathcal{J}_y$$

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\mu \frac{\partial \mathcal{H}_z}{\partial t} \qquad \frac{\partial \mathcal{H}_y}{\partial x} - \frac{\partial \mathcal{H}_x}{\partial y} = \varepsilon \frac{\partial \mathcal{E}_z}{\partial t} + \sigma \mathcal{E}_z + \mathcal{J}_z$$

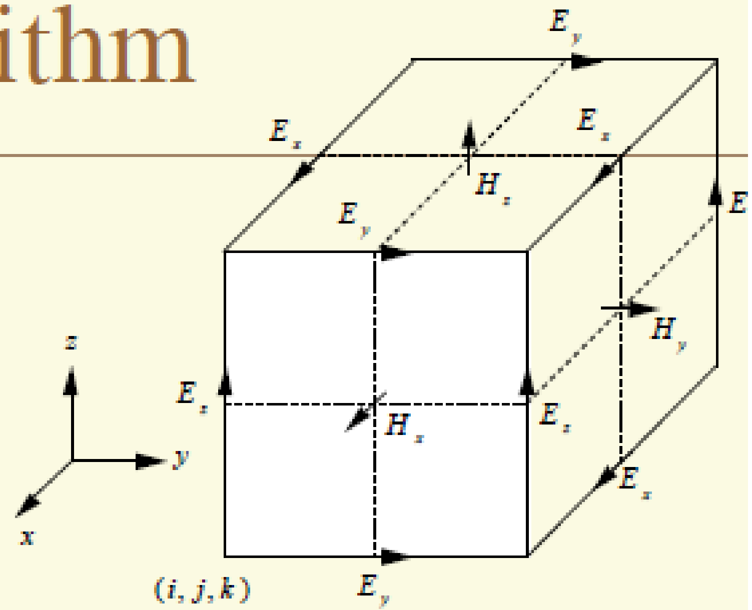
3-D FDTD Algorithm

FDTD discretization and Yee cell:





3-D FDTD Algorithm



FDTD discretization:

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial \mathcal{H}_x}{\partial t}$$

$$\frac{\mathcal{E}_z^n(i,j+1,k+\tfrac{1}{2}) - \mathcal{E}_z^n(i,j,k+\tfrac{1}{2})}{\Delta y}$$

$$-\frac{\mathcal{E}_y^n(i,j+\tfrac{1}{2},k+1) - \mathcal{E}_y^n(i,j+\tfrac{1}{2},k)}{\Delta z}$$

$$= -\mu \frac{\mathcal{H}_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) - \mathcal{H}_x^{n-\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2})}{\Delta t}$$

3-D FDTD Algorithm

Stability condition:

$$\Delta t \le \frac{1}{c\sqrt{\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}+\frac{1}{(\Delta z)^2}}}$$

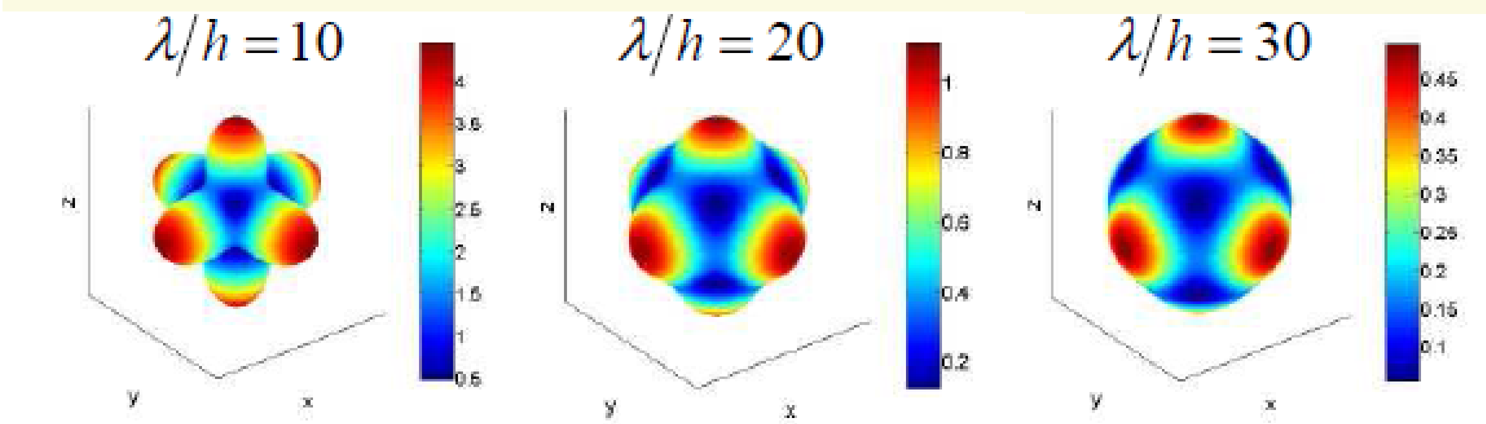
Numerical dispersion:

$$\frac{\tilde{k}-k}{k} \approx \frac{1}{24}\Big\{[(k\Delta x)^2\cos^4\phi^i + (k\Delta y)^2\sin^4\phi^i]\sin^4\theta^i + (k\Delta z)^2\cos^4\theta^i - (\omega\Delta t)^2\Big\}$$

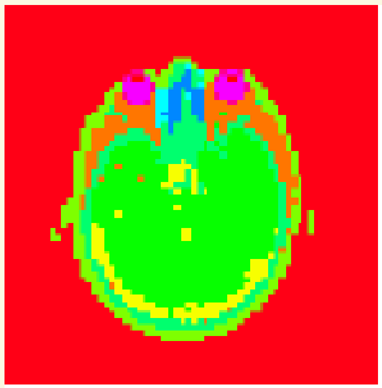
3-D FDTD Algorithm

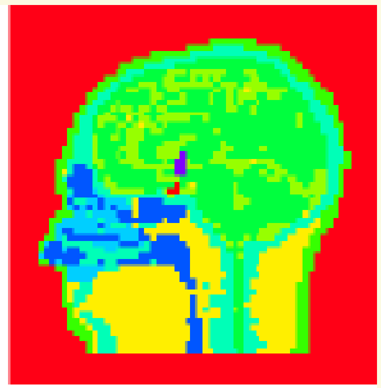
Numerical dispersion error for $\Delta t = 0.5 h/c$ **:**

$$\frac{\tilde{k}-k}{k} \approx \frac{\pi^2}{6}\left(\frac{h}{\lambda}\right)^2 \left\{[\cos^4\phi^i + \sin^4\phi^i]\sin^4\theta^i + \cos^4\theta^i - \tfrac{1}{4}\right\}$$



3-D FDTD Example





4 mm resolution

14 kinds of tissues

Outline

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- Finite Difference Time Domain method
- Finite element method
- Integral equation methods

Integral equation procedure

- Formulate an integral equation
- Expand the unknowns in terms of basis functions
- Define a discrete matrix representation of the continuous integral equation
- Solve the matrix equation
- Post-process to derive the required parameters

Advantages

- Only the structure of interest is discretized
- Outgoing conditions are satisfied automatically
- No numerical dispersion
- Can be formulated in time and frequency
- Can be defined for surfaces and volumes
- Can be formulated for complex backgrounds
- Can be higher order accurate

Challenges

- Computational cost of “simple” methods is $\sim O(N^2)$
- → Need for fast methods ($\sim O(N)$ or $O(N\log N)$) for N degrees of freedom
- Different formulations are required for different problem types, e.g. volumes vs. surfaces

Equation of electrostatics

- **Electrostatics**
  - $\partial/\partial t = \omega = 0 \Rightarrow$ **two uncoupled sets of (static) equations are obtained for** $\mathbf{E}\,\&\,\mathbf{D}$ **and** $\mathbf{H}\,\&\,\mathbf{B}$
  - **Equation of electrostatics**

$$\nabla \times \mathbf{E} = 0 \qquad \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

or

$$\nabla \cdot \mathbf{D} = \rho \qquad \oiint_S \mathbf{D} \cdot d\mathbf{s} = Q$$

  - **Boundary conditions**

$$\hat{\mathbf{n}}_{12} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0$$

$$\hat{\mathbf{n}}_{12} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_s$$

  - **Constitutive relation** $\mathbf{D} = \varepsilon \mathbf{E}$

Electric scalar potential

- Potential due to various charge distributions
  - Point charge $$V(\mathbf{R}) = \frac{q}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|}$$
  - Many discrete charges $$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{R} - \mathbf{R}_i|}$$
  - Various continuous charge distributions in free space

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\rho_v}{R'}\,dv' \quad \text{(volume distribution)}, \quad (4.48a)$$

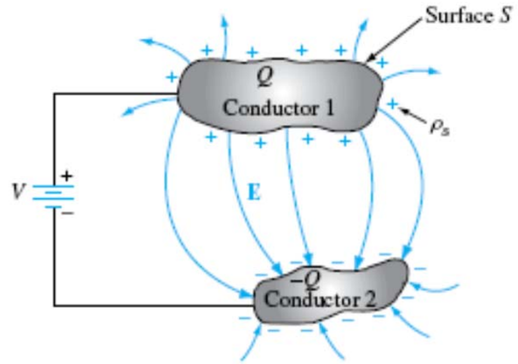
$$R' = |\mathbf{R} - \mathbf{R}_i|$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{S'}\frac{\rho_s}{R'}\,ds' \quad \text{(surface distribution)}, \quad (4.48b)$$

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{l'}\frac{\rho_l}{R'}\,dl' \quad \text{(line distribution)}. \quad (4.48c)$$

Capacitance

- Consider a capacitor consisting of two conductors with a voltage
- Opposite charges appear on the conductors
- Capacitance $C = \frac{Q}{V}$ (C/V or F).
- Alternative expression



$$Q = \int_S \rho_s\, ds = \int_S \varepsilon \hat{\mathbf{n}} \cdot \mathbf{E}\, ds = \int_S \varepsilon \mathbf{E} \cdot d\mathbf{s},$$

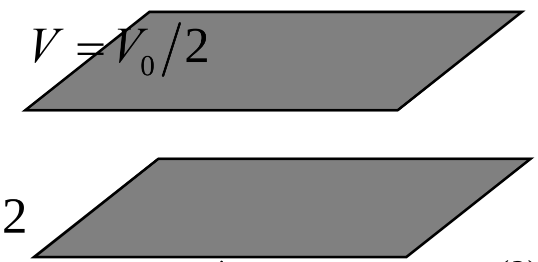
$$V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l},$$

$$\Rightarrow \quad C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}}$$

- Relation to resistance $RC = \frac{\varepsilon}{\sigma}$

Numerical capacitance extraction (1)

- Integral equation method
  - Given
    - Two conductors
    - Voltage on the conductors is $V_0/2$ and $-V_0/2$
    - Find the capacitance C



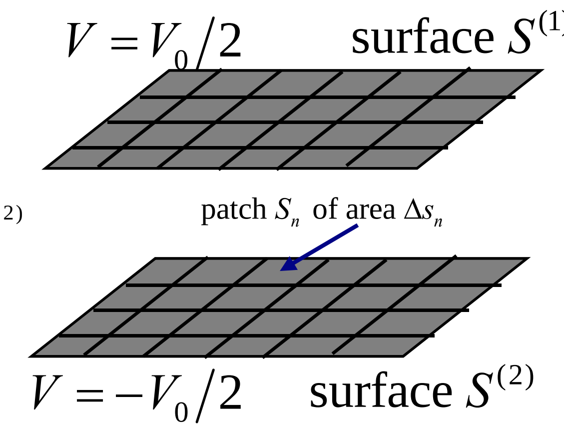
  - Formulation
    - Potential on the surfaces is given by $V_0/2$ and $-V_0/2$
    - Potential everywhere is calculated as $V(\mathbf{r}) = \int_S \underbrace{\frac{1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}}_{\text{Green's function}} \underbrace{\rho_s(\mathbf{r}')}_{\text{surface charge distribution}} dS'$ where ρ_s is an unknown surface charge
    - Equate the potentials on the surfaces

$$\Rightarrow \iint_{S^{(1)}+S^{(2)}} \underbrace{\frac{1}{4\pi\varepsilon_0 |\mathbf{r}-\mathbf{r}'|}}_{\text{Green's function}} \underbrace{\rho_s(\mathbf{r}')}_{\substack{\text{surface charge}\\\text{distribution}\\\text{(unknown)}}} dS' = \begin{cases} V_0/2; \mathbf{r} \in S_1 \\ -V_0/2; \mathbf{r} \in S_2 \end{cases}$$

← integral equation

Numerical capacitance extraction (2)

- Matrix equation
  - Discretize the problem
    - Divide the surfaces into $N = N_{S^{(1)}} + N_{S^{(2)}}$ small patches, $N_{S^{(1)}}, N_{S^{(2)}}$ are numbers of patches on surfaces $S^{(1)}, S^{(2)}$
    - Replace the integration by summation
    - Enforce the IE at the location of charges
    - $$\Rightarrow \sum_{n=1}^{N} \iint_{S_n} \frac{\rho_s(\mathbf{r}')ds'}{4\pi\varepsilon_0 |\mathbf{r}_m - \mathbf{r}'|} = \begin{cases} V_0/2; & \mathbf{r}_m \in S_1 \\ -V_0/2; & \mathbf{r}_m \in S_2 \end{cases}$$



  - Obtain a matrix equation

$$\sum_{n=1}^{N} \iint_{S_n} \frac{\rho_s(\mathbf{r}')ds'}{4\pi\varepsilon_0 |\mathbf{r}_m - \mathbf{r}'|} = \begin{cases} V_0/2_0; & \mathbf{r}_m \in S_1 \\ -V_0/2; & \mathbf{r}_m \in S_2 \end{cases} \Rightarrow \sum_{n=1}^{N} Z_{mn} Q_n = V_m \Rightarrow \boxed{\underbrace{\underline{\underline{Z}}\,\underline{Q} = \underline{V}}_{\text{matrix equation}}}$$

Numerical capacitance extraction (3)

- Matrix equation and solution
  - Matrix equation $\underline{\underline{Z}}\,\underline{Q} = \underline{V}$

$$\underline{\underline{Z}} : N \times N \text{ matrix}; Z_{mn} = \frac{1}{\Delta s_n} \iint_{S_n} \frac{ds'}{4\pi\varepsilon_0 |\mathbf{r}_m - \mathbf{r}'|} \Rightarrow \begin{cases} Z_{mn} \approx \dfrac{1}{4\pi\varepsilon_0 |\mathbf{r}_m - \mathbf{r}_n|}; \ \mathbf{r}_m \neq \mathbf{r}_n \\ Z_{mn} \approx \dfrac{1}{2\varepsilon_0 \sqrt{\pi\,\Delta s_n}}; \ \mathbf{r}_m = \mathbf{r}_n \end{cases}$$

$$\underline{Q} : N \text{ vector}; \ Q_n = \rho_s(\mathbf{r}_n)\Delta s_n$$

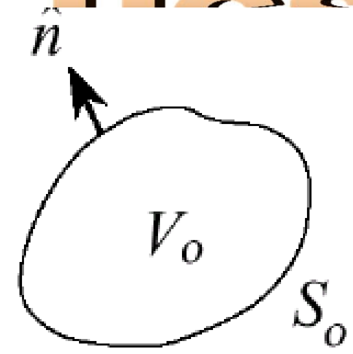
$$\underline{V} : N \text{ vector}; \ V_m = \begin{cases} V_0/2; \ \mathbf{r}_m \in S_1 \\ -V_0/2; \ \mathbf{r}_m \in S_2 \end{cases}$$

  - Solution $\underline{Q} = \underline{\underline{Z}}^{-1}\underline{V}$

- Extracted capacitance

$$C = \frac{\overbrace{Q_{S^{(1)}}}^{\text{total charge on surface 1}}}{V_0} \Rightarrow \boxed{C = \frac{\sum_{n=1}^{N_{S^{(1)}}} Q_n}{V_0}}$$

3D Integral Equations



Electric field integral equation (EFIE):

$$\frac{1}{2}\mathbf{M}_s(\mathbf{r}) - \hat{n}\times\mathcal{L}(\bar{\mathbf{J}}_s) + \hat{n}\times\tilde{\mathcal{K}}(\mathbf{M}_s) = -\hat{n}\times\mathbf{E}^{\text{inc}}(\mathbf{r}) \qquad \mathbf{r}\in S_o$$

Magnetic field integral equation (MFIE):

$$\frac{1}{2}\bar{\mathbf{J}}_s + \hat{n}\times\mathcal{L}(\mathbf{M}_s) + \hat{n}\times\tilde{\mathcal{K}}(\bar{\mathbf{J}}_s) = \hat{n}\times\bar{\mathbf{H}}^{\text{inc}}(\mathbf{r}) \qquad \mathbf{r}\in S_o$$

For a conducting object: $\hat{n}\times\mathbf{E}(\mathbf{r}) = 0 \quad \mathbf{r}\in S_o \Rightarrow \mathbf{M}_s = 0$

EFIE: $$\hat{n}\times\mathcal{L}(\bar{\mathbf{J}}_s) = \hat{n}\times\mathbf{E}^{\text{inc}}(\mathbf{r}) \qquad \mathbf{r}\in S_o$$

MFIE: $$\frac{1}{2}\bar{\mathbf{J}}_s + \hat{n}\times\tilde{\mathcal{K}}(\bar{\mathbf{J}}_s) = \hat{n}\times\bar{\mathbf{H}}^{\text{inc}}(\mathbf{r}) \qquad \mathbf{r}\in S_o$$

3D Integral Equations

Introduce operators:

$$\mathcal{L}(\mathbf{X}) = jk_0 \oiint_{S_o} \left[\mathbf{X}(\mathbf{r}')G_0(\mathbf{r},\mathbf{r}') + \frac{1}{k_0^2} \nabla' \cdot \mathbf{X}(\mathbf{r}') \nabla G_0(\mathbf{r},\mathbf{r}') \right] dS'$$

$$\mathcal{K}(\mathbf{X}) = \oiint_{S_o} \mathbf{X}(\mathbf{r}') \times \nabla G_0(\mathbf{r},\mathbf{r}') dS'$$

Introduce equivalent currents:

$$\overline{\mathbf{J}}_s(\mathbf{r}') = \hat{n}' \times \overline{\mathbf{H}}(\mathbf{r}') = Z_0 \hat{n}' \times \mathbf{H}(\mathbf{r}') \qquad \mathbf{M}_s(\mathbf{r}') = \mathbf{E}(\mathbf{r}') \times \hat{n}'$$

$$\mathbf{E}^{\text{inc}}(\mathbf{r}) - \mathcal{L}(\overline{\mathbf{J}}_s) + \mathcal{K}(\mathbf{M}_s) = \begin{cases} \mathbf{E}(\mathbf{r}) & \mathbf{r} \in V_\infty \\ 0 & \mathbf{r} \in V_o \end{cases}$$

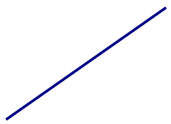
$$\overline{\mathbf{H}}^{\text{inc}}(\mathbf{r}) - \mathcal{L}(\mathbf{M}_s) - \mathcal{K}(\overline{\mathbf{J}}_s) = \begin{cases} \overline{\mathbf{H}}(\mathbf{r}) & \mathbf{r} \in V_\infty \\ 0 & \mathbf{r} \in V_o \end{cases}$$

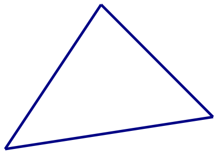
Outline

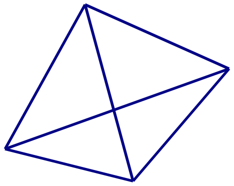
- Introduction
- Maxwell's equations
- Finite Difference Time Domain method
- Finite element method
- Integral equation methods

Basic FEM Steps

1. Discretization/subdivision of solution domain

1-D: 

2-D: 

3-D: 

2. Selection of interpolation schemes

Linear or higher-order polynomials

3. Formulation of the system of equations

Using either the Ritz or Galerkin method:
Formulate elemental equations and assemble

4. Solution of the system of equations

Using either a direct or iterative method

Advantages

- Works for general structures
- Can be formulated in time and frequency
- Can be higher order accurate
- Low cost (~O(N))

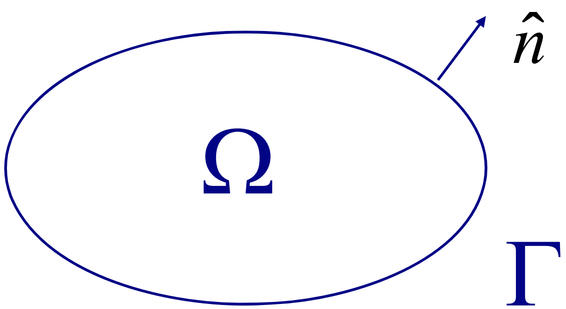
Boundary-Value Problem

Partial differential equation (PDE):

$$-\nabla \cdot (\alpha \nabla \phi) + \beta \phi = f \qquad (x, y) \in \Omega$$

Boundary condition (BC):

$$\phi = p \qquad \text{on } \Gamma$$



Equivalent Variational Problem

Equivalent variational problem:

$$\delta F(\phi) = 0$$

where

$$F(\phi) = \frac{1}{2}\iint_\Omega \left[\alpha \nabla\phi \cdot \nabla\phi + \beta\phi^2\right] d\Omega - \iint_\Omega f\phi \, d\Omega$$

Equivalent Variational Problem

Proof:

Taking the first variation:

$$\delta F(\phi) = \iint_\Omega [\alpha \nabla \phi \cdot \nabla \delta \phi + \beta \phi \delta \phi]\, d\Omega - \iint_\Omega f \delta \phi \; d\Omega$$

Invoking the identity

$$\alpha \nabla \phi \cdot \nabla \delta \phi = \nabla \cdot (\alpha \delta \phi \nabla \phi) - \delta \phi \nabla \cdot (\alpha \nabla \phi)$$

$$\delta F(\phi) = \iint_\Omega \nabla \cdot [\alpha \delta \phi \nabla \phi]\, d\Omega + \iint_\Omega [-\nabla \cdot (\alpha \nabla \phi) + \beta \phi]\, \delta \phi d\Omega$$
$$- \iint_\Omega f \delta \phi \; d\Omega$$

Equivalent Variational Problem

Applying the divergence theorem

$$\iint_\Omega \nabla \cdot \mathbf{a}\, d\Omega = \oint_\Gamma \mathbf{a} \cdot \hat{n}\, d\Gamma$$

$$\delta F(\phi) = \iint_\Omega \left[-\nabla \cdot (\alpha \nabla \phi) + \beta\phi - f\right] \delta\phi\, d\Omega + \oint_\Gamma \alpha \frac{\partial \phi}{\partial n} \delta\phi\, d\Gamma$$

Equivalent Variational Problem

Since $\phi = p$ on Γ, $\delta\phi = 0$ on Γ. Therefore,

$$\delta F(\phi) = \iint_\Omega \left[-\nabla \cdot (\alpha \nabla \phi) + \beta\phi - f \right] \delta\phi \, d\Omega$$

Imposing $\delta F(\phi) = 0$, we obtain

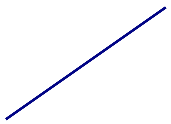
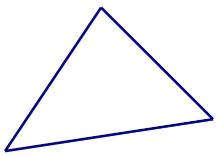
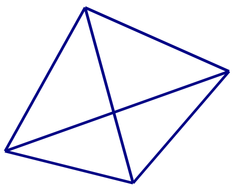
$$\iint_\Omega \left[-\nabla \cdot (\alpha \nabla \phi) + \beta\phi - f \right] \delta\phi \, d\Omega = 0$$

Because $\delta\phi$ is an arbitrary variation,

$$-\nabla \cdot (\alpha \nabla \phi) + \beta\phi - f = 0$$

Basic FEM Steps

1. Discretization/subdivision of solution domain

1-D:  2-D:  3-D: 

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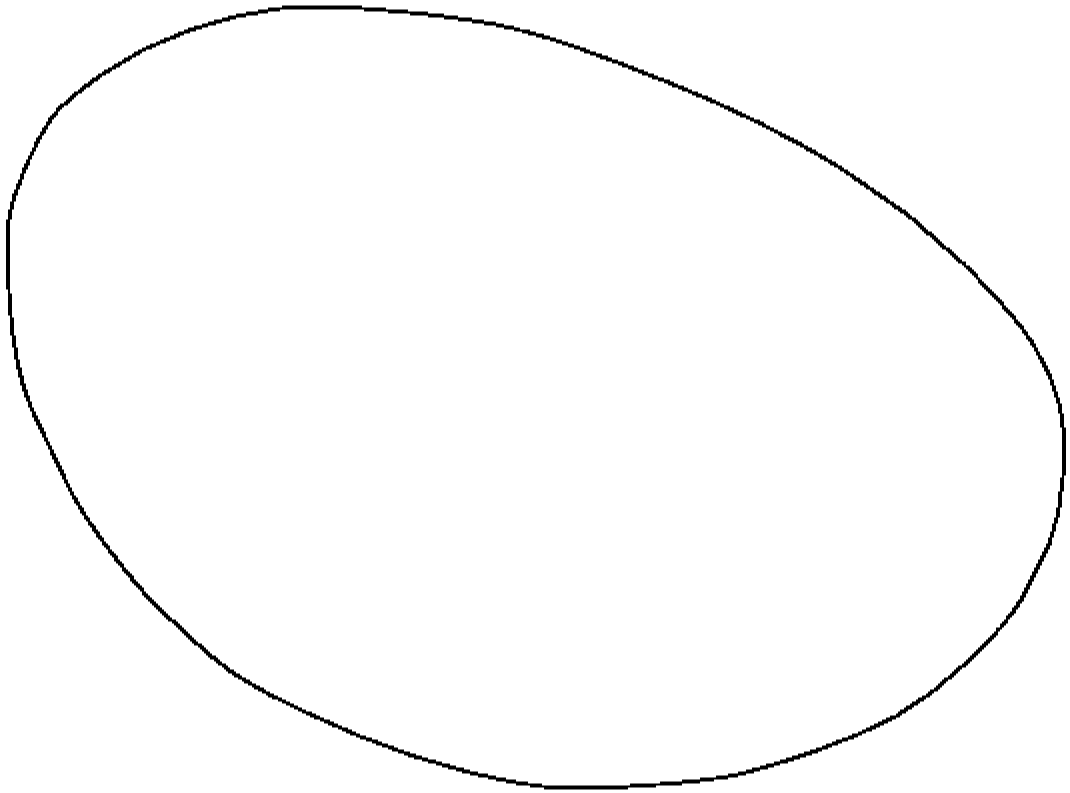
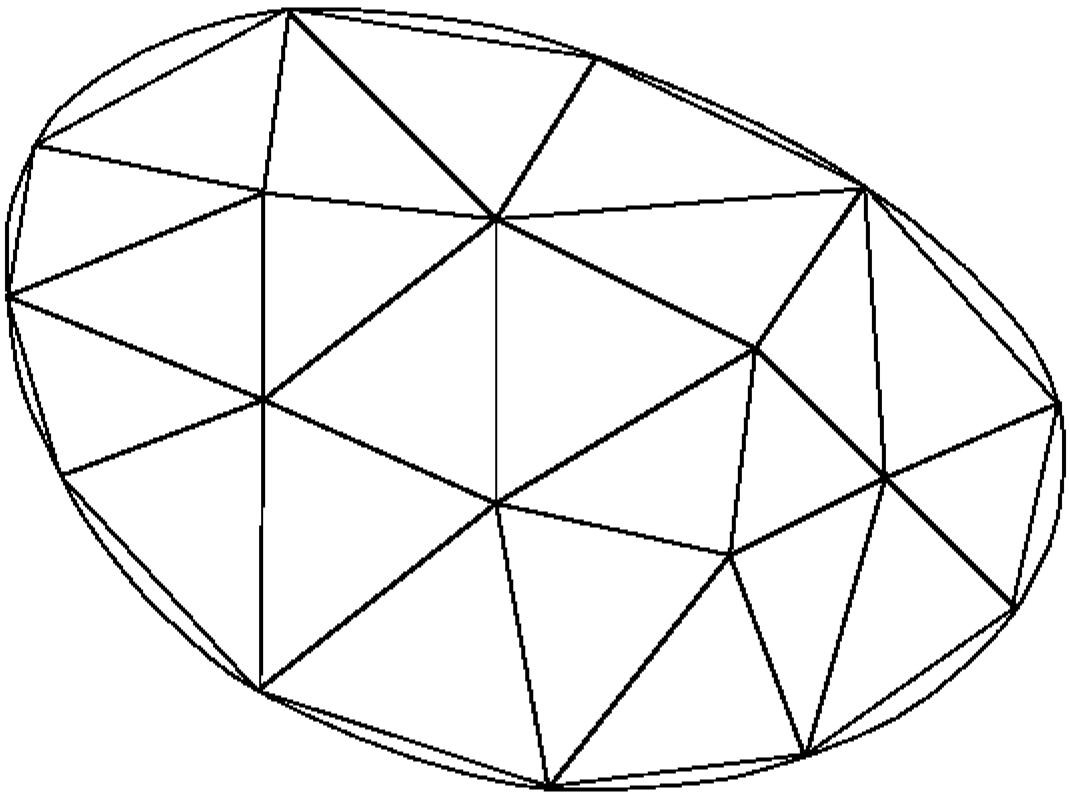
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



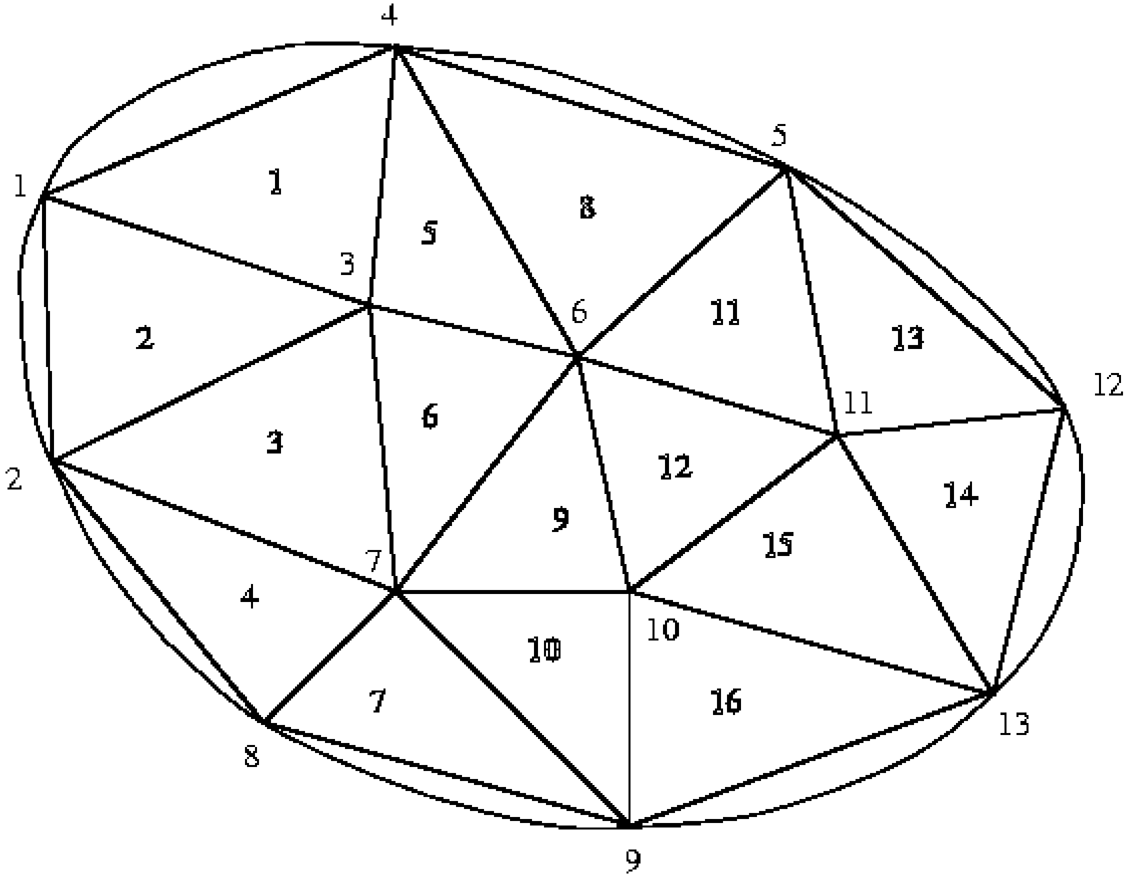
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



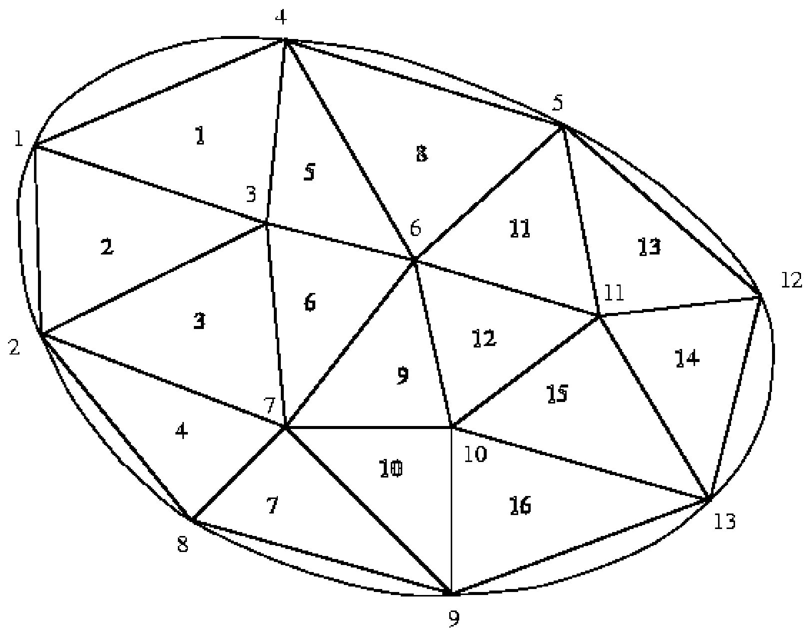
FEM Analysis – Domain subdivision

Step 1: Domain Discretization



FEM Analysis – Domain subdivision

Step 1: Domain Discretization



e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	1	3	4
2	2	3	1
3	2	7	3
4	8	7	2
5	3	6	4
6	7	6	3
7	8	9	7
8	6	5	4
9	7	10	6
10	9	10	7
⋮	⋮	⋮	⋮

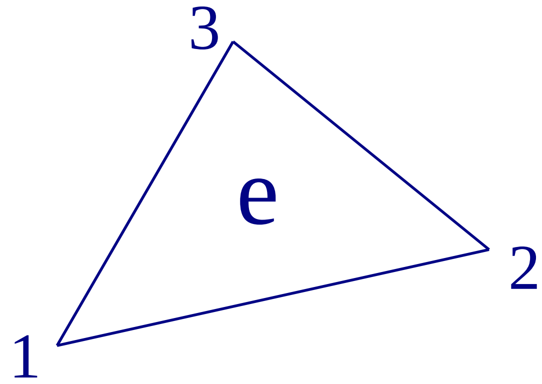
FEM Analysis – Element interpolation

Step 2: Element Interpolation

Assume

$$\phi^e(x,y) = a^e + b^e x + c^e y$$

Apply at the three nodes:



$$\phi_1^e = a^e + b^e x_1^e + c^e y_1^e$$

$$\phi_2^e = a^e + b^e x_2^e + c^e y_2^e$$

$$\phi_3^e = a^e + b^e x_3^e + c^e y_3^e$$

Solve for a^e, b^e, and c^e.

FEM Analysis – Element interpolation

$$\phi^e(x,y) = N_1^e(x,y)\phi_1^e + N_2^e(x,y)\phi_2^e + N_3^e(x,y)\phi_3^e$$

$$= \sum_{j=1}^{3} N_j^e(x,y)\phi_j^e$$

$$N_j^e(x,y) = \frac{1}{2\Delta^e}(a_j^e + b_j^e x + c_j^e y) \qquad j = 1,2,3.$$

$$a_1^e = x_2^e y_3^e - y_2^e x_3^e \; ; \quad b_1^e = y_2^e - y_3^e \; ; \quad c_1^e = x_3^e - x_2^e$$

$$a_2^e = x_3^e y_1^e - y_3^e x_1^e \; ; \quad b_2^e = y_3^e - y_1^e \; ; \quad c_2^e = x_1^e - x_3^e$$

$$a_3^e = x_1^e y_2^e - y_1^e x_2^e \; ; \quad b_3^e = y_1^e - y_2^e \; ; \quad c_3^e = x_2^e - x_1^e$$

Δ^e = area of the eth element

FEM Analysis – Element formulation

Step 3: Formulation of the System of Equations

A. Elemental equations

$$F(\phi) = \sum_{e=1}^{M} F^e(\phi^e)$$

$$F^e(\phi^e) = \frac{1}{2}\iint_{\Omega^e} \left[\alpha \nabla\phi^e \cdot \nabla\phi^e + \beta\left(\phi^e\right)^2\right] d\Omega - \iint_{\Omega^e} f\,\phi^e\, d\Omega$$

FEM Analysis – Element formulation

Elemental functional:

$$F^e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\phi_i^e\phi_j^e \iint_{\Omega^e}\left(\alpha\nabla N_i^e\cdot\nabla N_j^e + \beta N_i^e N_j^e\right)d\Omega$$

$$-\sum_{i=1}^{3}\phi_i^e \iint_{\Omega^e} f N_i^e\, d\Omega$$

$$F^e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\phi_i^e K_{ij}^e\phi_j^e - \sum_{i=1}^{3}\phi_i^e b_i^e$$

FEM Analysis – Element formulation

$$K_{ij}^e = \iint_{\Omega^e} \left(\alpha \nabla N_i^e \cdot \nabla N_j^e + \beta N_i^e N_j^e\right) dx dy \qquad i,j = 1,2,3$$

$$b_i^e = \iint_{\Omega^e} f N_i^e \, dx dy \qquad i = 1,2,3.$$

Integration formula:

$$\iint_{\Omega^e} \left(N_1^e\right)^l \left(N_2^e\right)^m \left(N_3^e\right)^n dx dy = \frac{l!m!n!}{(l+m+n+2)!} 2\Delta^e$$

$$K_{ij}^e = \frac{\alpha^e}{4\Delta^e}\left(b_i^e b_j^e + c_i^e c_j^e\right) + \frac{\Delta^e}{12}\beta^e\left(1+\delta_{ij}\right)$$

$$b_i^e = \frac{\Delta^e}{3} f^e.$$

FEM Analysis – Element formulation

Use matrix notation:

$$F^e = \frac{1}{2}\{\phi^e\}^T[K^e]\{\phi^e\} - \{\phi^e\}^T\{b^e\}$$

$$[K^e] = \begin{bmatrix} K_{11}^e & K_{12}^e & K_{13}^e \\ K_{21}^e & K_{22}^e & K_{23}^e \\ K_{31}^e & K_{32}^e & K_{33}^e \end{bmatrix} ; \quad \{\phi^e\} = \begin{Bmatrix} \phi_1^e \\ \phi_2^e \\ \phi_3^e \end{Bmatrix} ; \quad \{b^e\} = \begin{Bmatrix} b_1^e \\ b_2^e \\ b_3^e \end{Bmatrix} .$$

FEM Analysis – Assembly

B. Assembly

$$F^e = \frac{1}{2}\{\phi\}^T[\bar{K}^e]\{\phi\} - \{\phi\}^T\{\bar{b}^e\}$$

$$[K^e] = \begin{bmatrix} K^e_{11} & K^e_{12} & K^e_{13} \\ K^e_{21} & K^e_{22} & K^e_{23} \\ K^e_{31} & K^e_{32} & K^e_{33} \end{bmatrix} \Rightarrow [\bar{K}^{(1)}] = \begin{bmatrix} K^{(1)}_{33} & K^{(1)}_{31} & 0 & K^{(1)}_{32} & 0 & 0 \\ K^{(1)}_{13} & K^{(1)}_{11} & 0 & K^{(1)}_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ K^{(1)}_{23} & K^{(1)}_{21} & 0 & K^{(1)}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

FEM Analysis – Assembly

Carry out the summation:

$$F = \frac{1}{2}\sum_{e=1}^{M}\{\phi\}^T[\bar{K}^e]\{\phi\} - \sum_{e=1}^{M}\{\phi\}^T\{\bar{b}^e\}$$

$$= \frac{1}{2}\{\phi\}^T\left(\sum_{e=1}^{M}[\bar{K}^e]\right)\{\phi\} - \{\phi\}^T\left(\sum_{e=1}^{M}\{\bar{b}^e\}\right)$$

$$= \frac{1}{2}\{\phi\}^T[K]\{\phi\} - \{\phi\}^T\{b\}$$

Apply the stationarity condition:

$$\delta F = 0 \quad \Rightarrow \quad [K]\{\phi\} = \{b\}$$

FEM Analysis – Assembly

How to carry out the summation?

$$[K] = \sum_{e=1}^{M} \left[\bar{K}^e\right], \quad \{b\} = \sum_{e=1}^{M} \left\{\bar{b}^e\right\}$$

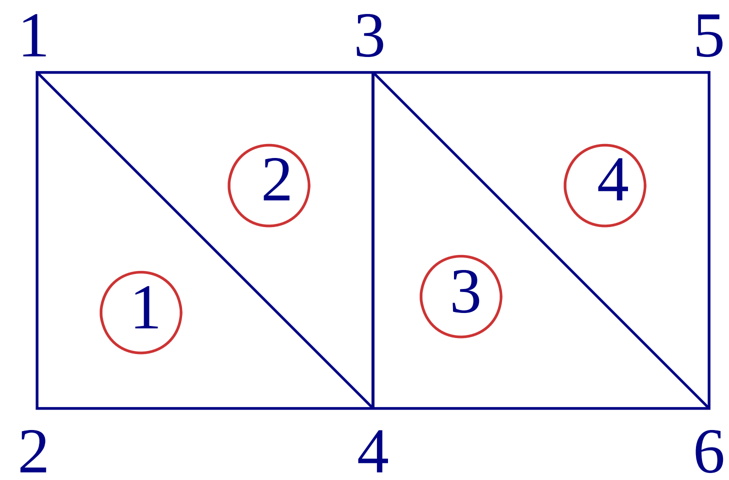
To assemble $[K]$: Add each K^e_{ij} to $K_{n(i,e),n(j,e)}$.

To assemble $\{b\}$: Add each b^e_i to $b_{n(i,e)}$.

FEM Analysis – Assembly

Example:

$$\{\phi\} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{Bmatrix}$$



e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis

1. Start from a null matrix and add in the first element:

$$[K] = \begin{bmatrix} \boxed{K_{33}^{(1)}} & \boxed{K_{31}^{(1)}} & 0 & \boxed{K_{32}^{(1)}} & 0 & 0 \\ \boxed{K_{13}^{(1)}} & \boxed{K_{11}^{(1)}} & 0 & \boxed{K_{12}^{(1)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \boxed{K_{23}^{(1)}} & \boxed{K_{21}^{(1)}} & 0 & \boxed{K_{22}^{(1)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

FEM Analysis

2. Add in the second element:

$$[K] = \begin{bmatrix} K_{33}^{(1)} + \boxed{K_{33}^{(2)}} & K_{31}^{(1)} & \boxed{K_{32}^{(2)}} & K_{32}^{(1)} + \boxed{K_{31}^{(2)}} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ \boxed{K_{23}^{(2)}} & 0 & \boxed{K_{22}^{(2)}} & \boxed{K_{21}^{(2)}} & 0 & 0 \\ K_{23}^{(1)} + \boxed{K_{13}^{(2)}} & K_{21}^{(1)} & \boxed{K_{12}^{(2)}} & K_{22}^{(1)} + \boxed{K_{11}^{(2)}} & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

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3. Add in the third element:

e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

$$[K] = \begin{bmatrix} K_{33}^{(1)} + K_{33}^{(2)} & K_{31}^{(1)} & K_{32}^{(2)} & K_{32}^{(1)} + K_{31}^{(2)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ K_{23}^{(2)} & 0 & K_{22}^{(2)} + \boxed{K_{33}^{(3)}} & K_{21}^{(2)} + \boxed{K_{31}^{(3)}} & 0 & \boxed{K_{32}^{(3)}} \\ K_{23}^{(1)} + K_{13}^{(2)} & K_{21}^{(1)} & K_{12}^{(2)} + \boxed{K_{12}^{(3)}} & K_{22}^{(1)} + K_{11}^{(2)} + \boxed{K_{11}^{(3)}} & 0 & \boxed{K_{12}^{(3)}} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{K_{23}^{(3)}} & \boxed{K_{21}^{(3)}} & 0 & \boxed{K_{22}^{(3)}} \end{bmatrix}$$

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e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

4. Add in the fourth element:

$$[K] =$$

$$\begin{bmatrix} K_{33}^{(1)}+K_{33}^{(2)} & K_{31}^{(1)} & K_{32}^{(2)} & K_{32}^{(1)}+K_{31}^{(2)} & 0 & 0 \\ K_{13}^{(1)} & K_{11}^{(1)} & 0 & K_{12}^{(1)} & 0 & 0 \\ K_{23}^{(2)} & 0 & K_{22}^{(2)}+K_{33}^{(3)}+\boxed{K_{33}^{(4)}} & K_{21}^{(2)}+K_{31}^{(3)} & \boxed{K_{32}^{(4)}} & K_{32}^{(3)}+\boxed{K_{31}^{(4)}} \\ K_{23}^{(1)}+K_{13}^{(2)} & K_{21}^{(1)} & K_{12}^{(2)}+K_{12}^{(3)} & K_{22}^{(1)}+K_{11}^{(2)}+K_{11}^{(3)} & 0 & K_{12}^{(3)} \\ 0 & 0 & \boxed{K_{23}^{(4)}} & 0 & \boxed{K_{22}^{(4)}} & \boxed{K_{21}^{(4)}} \\ 0 & 0 & K_{23}^{(3)}+\boxed{K_{13}^{(4)}} & K_{21}^{(3)} & \boxed{K_{12}^{(4)}} & K_{22}^{(3)}+\boxed{K_{11}^{(4)}} \end{bmatrix}$$

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e	$n(1,e)$	$n(2,e)$	$n(3,e)$
1	2	4	1
2	4	3	1
3	4	6	3
4	6	5	3

5. Follow a similar procedure:

$$\{b\} = \left\{ \begin{array}{c} b_3^{(1)} + b_3^{(2)} \\ b_1^{(1)} \\ b_2^{(2)} + b_3^{(3)} + b_3^{(4)} \\ b_2^{(1)} + b_1^{(2)} + b_1^{(3)} \\ b_2^{(4)} \\ b_2^{(3)} + b_1^{(4)} \end{array} \right\}$$

FEM Analysis – Apply BC

C. Impose the Dirichlet Boundary Condition:

- Approach #1:

To impose $\phi_1 = p_1$, simply set:

$$K_{11} = 1, \quad K_{1i} = 0 \quad \text{for} \quad i = 2, 3, 4, 5, 6, \quad b_1 = p_1$$

To maintain symmetry, set:

$$b_i \leftarrow b_i - K_{i1} p_1, \quad K_{i1} = 0 \quad \text{for} \quad i = 2, 3, 4, 5, 6$$

FEM Analysis – Apply BC

After imposing $\phi_1 = p_1$, $\phi_3 = p_3$, $\phi_5 = p_5$:

$$[K] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & 0 & K_{24} & 0 & K_{26} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & K_{42} & 0 & K_{44} & 0 & K_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & K_{62} & 0 & K_{64} & 0 & K_{66} \end{bmatrix}$$

Remains Symmetric!

$$\{b\} = \left\{ \begin{array}{c} p_1 \\ b_2 - K_{21}p_1 - K_{23}p_3 - K_{25}p_5 \\ p_3 \\ b_4 - K_{41}p_1 - K_{43}p_3 - K_{45}p_5 \\ p_5 \\ b_6 - K_{61}p_1 - K_{63}p_3 - K_{65}p_5 \end{array} \right\}$$

FEM Analysis – Apply BC

Can be made smaller:

$$\begin{bmatrix} K_{22} & K_{24} & K_{26} \\ K_{42} & K_{44} & K_{46} \\ K_{62} & K_{64} & K_{66} \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_4 \\ \phi_6 \end{Bmatrix}$$

$$= \begin{Bmatrix} b_2 - K_{21}p_1 - K_{23}p_3 - K_{25}p_5 \\ b_4 - K_{41}p_1 - K_{43}p_3 - K_{45}p_5 \\ b_6 - K_{61}p_1 - K_{63}p_3 - K_{65}p_5 \end{Bmatrix}$$

Worthwhile when there are many prescribed nodes.

FEM Analysis – Apply BC

- Approach #2 (Simple one):

To impose $\phi_1 = p_1$, simply set:

$$10^{70}\phi_1 + K_{12}\phi_2 + K_{13}\phi_3 + K_{14}\phi_4 + K_{15}\phi_5 + K_{16}\phi_6 = p_1 \times 10^{70}$$

After imposing $\phi_1 = p_1$, $\phi_3 = p_3$, $\phi_5 = p_5$:

$$\begin{bmatrix} 10^{70} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & 10^{70} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & 10^{70} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{Bmatrix} = \begin{Bmatrix} p_1 \times 10^{70} \\ b_2 \\ p_3 \times 10^{70} \\ b_4 \\ p_5 \times 10^{70} \\ b_6 \end{Bmatrix}$$