Theory of Optical Modes in Step Index Fibers

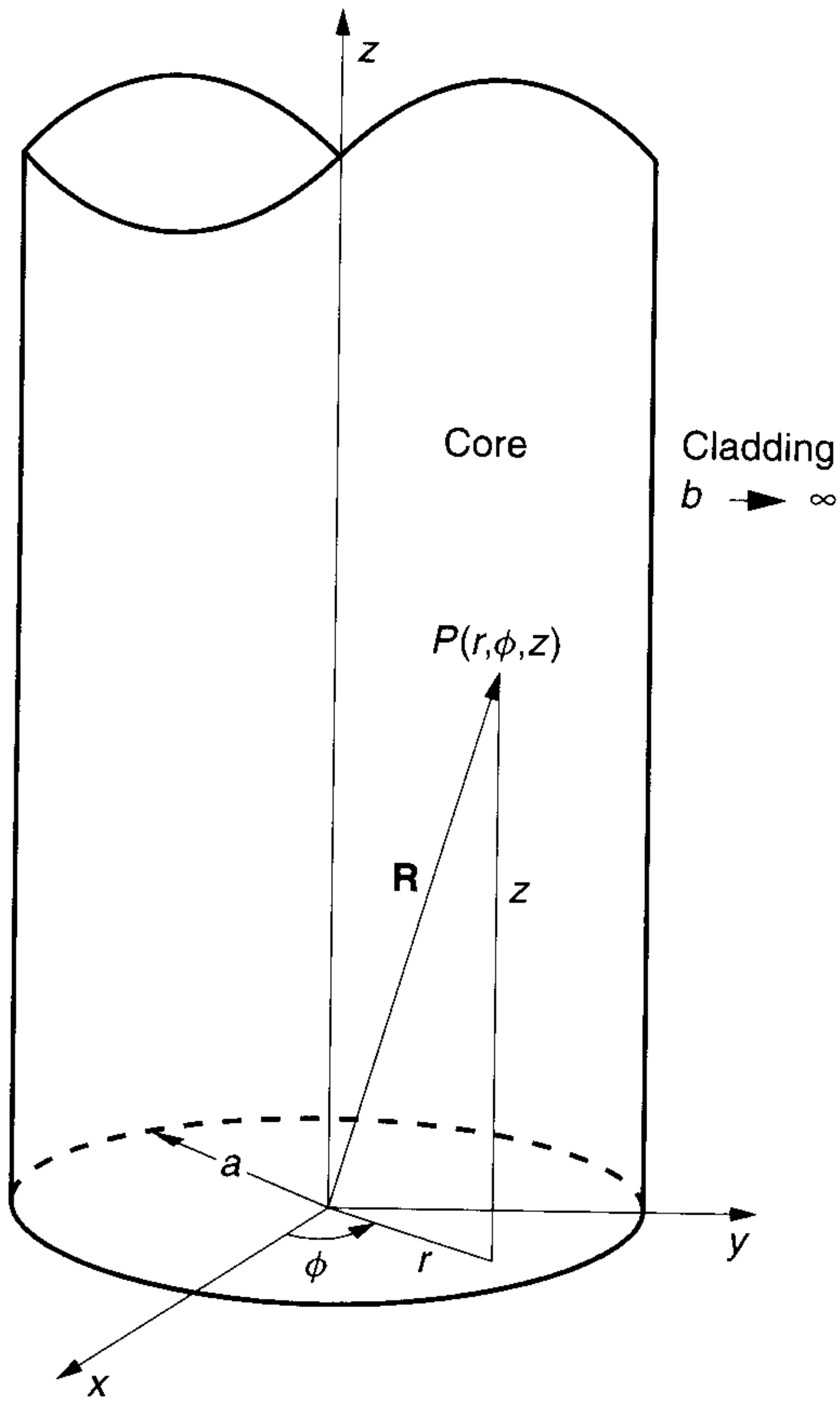


Figure 11.13 Optical fiber in a cylindrical coordinate system.

$$n = \begin{cases} n_{core} & \text{inside the core} \\ n_{clad} & \text{inside the cladding} \end{cases}$$

$$\vec{E} = E_r\hat{r} + E_\phi\hat{\phi} + E_z\hat{z}$$
$$\vec{H} = H_r\hat{r} + H_\phi\hat{\phi} + H_z\hat{z}$$

We find the modes by looking for solutions of:

$$\begin{aligned}\nabla^2\vec{E} + (nk_0)^2\vec{E} = &\left(\nabla^2 E_r - \frac{2}{r^2}\frac{\partial E_\phi}{\partial \phi} - \frac{E_r}{r^2} + (nk_0)^2 E_r\right)\hat{r} \\ &+ \left(\nabla^2 E_\phi + \frac{2}{r^2}\frac{\partial E_r}{\partial \phi} - \frac{E_\phi}{r^2} + (nk_0)^2 E_\phi\right)\hat{\phi} \\ &+ \left(\nabla^2 E_z + (nk_0)^2 E_z\right)\hat{z} = 0\end{aligned}$$

The equations have a simple physical interpretation.

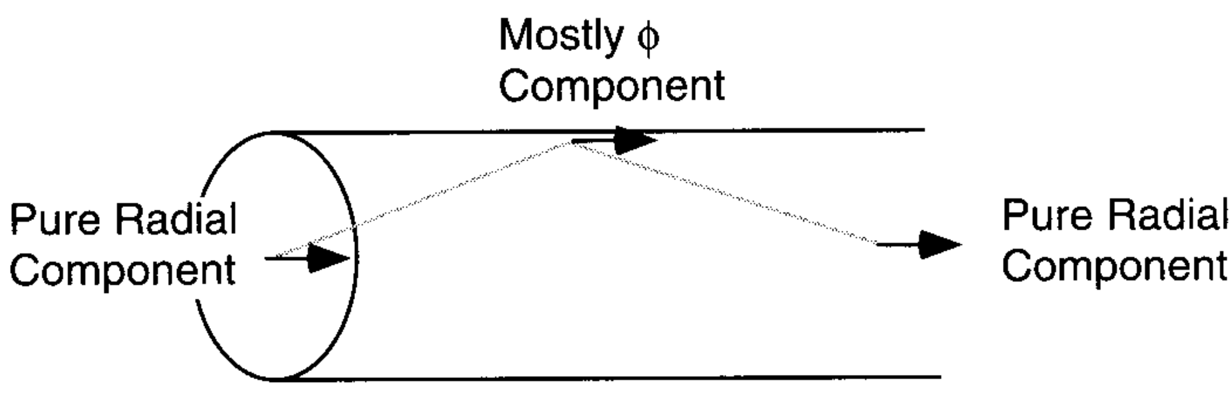


Figure 5.2 A radial field at one point in the waveguide will become an azimuthal field at another location. Notice that the field is not converted between the components by reflection, but by propagation through the coordinate system. *(from Pollock)*

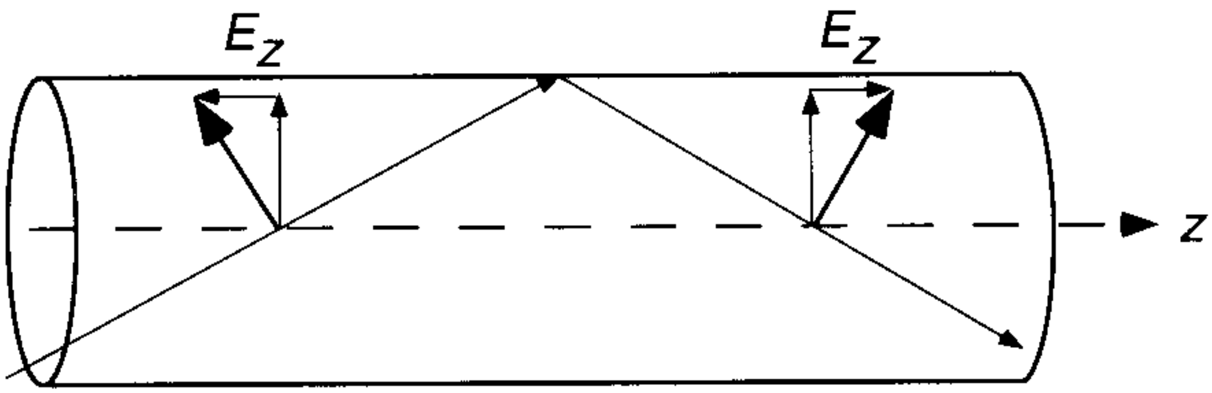


Figure 5.3 The longitudinal component of the electric field does not change through either propagation or reflection at the cylindrical surface. *(from Pollock)*

Since the equations for E_r and E_ϕ are coupled, we first solve for E_z. H_z is a solution of the same Helmholtz equation and its solutions have the same form. We find all other field components from E_z and H_z using Maxwell's equations.

We look for solutions of the form:

$$E_z = R(r)\Phi(\phi)Z(z)$$

In the core we find:

$$Z(z) = ae^{-j\beta z} + be^{j\beta z}$$
$$\Phi(\phi) = ce^{j\nu\phi} + de^{-j\nu\phi}$$
$$R(r) = eJ_\nu(\kappa r) + fN_\nu(\kappa r)$$

where $\kappa^2 = (n_{core}k_0)^2 - \beta^2$, and $\nu = 0,1,2...$

We can simplify these noting that:

- Often we have only forward going waves (b=0)
- The $N_\nu(\kappa r)$ solution goes to minus infinity at r = 0 so it is unphysical (f=0)

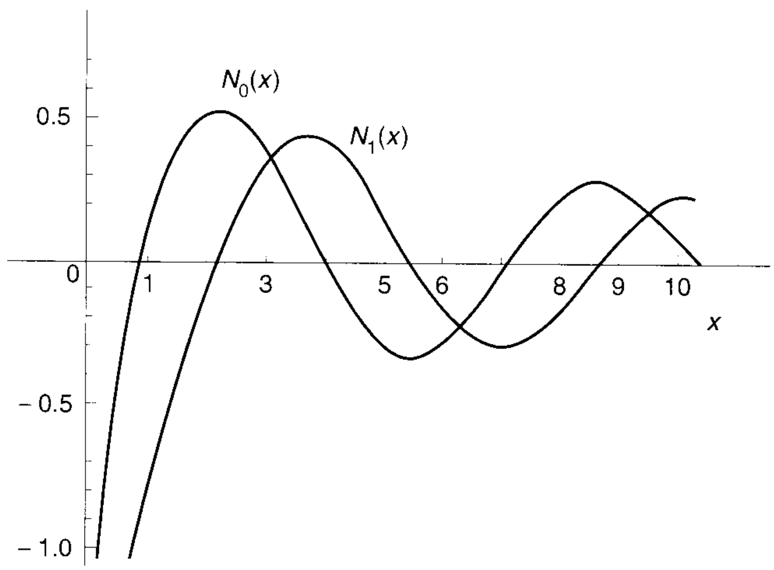


Figure 11.14 Zero- and first-order Bessel functions of the second kind.

- We need both the $e^{j\nu\phi}$ and $e^{-j\nu\phi}$ terms to describe the ϕ dependence of the eigenmodes, but we can limit the discussion to the $e^{j\nu\phi}$ solution with the understanding that a mode with $e^{-j\nu\phi}$ dependence can be found from the $e^{j\nu\phi}$ mode by rotating the fiber.

Then we can write:

$$\boxed{\begin{aligned} E_z &= AJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c. \\ H_z &= BJ_\nu(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c. \end{aligned}}$$

In the cladding region we find:

$$\boxed{\begin{aligned} E_z &= CK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c. \\ H_z &= DK_\nu(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c. \end{aligned}}$$

$$\text{where } -\gamma^2 = (n_{clad}k_0)^2 - \beta^2$$

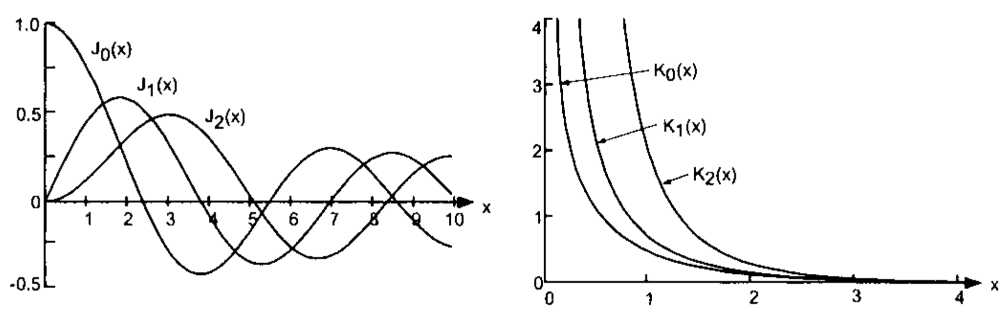


Figure 4.4. The first three Bessel functions of the first kind, $J_\nu(\kappa r)$, and of the second kind, $K_\nu(\gamma r)$.

From Pollock and Lipson

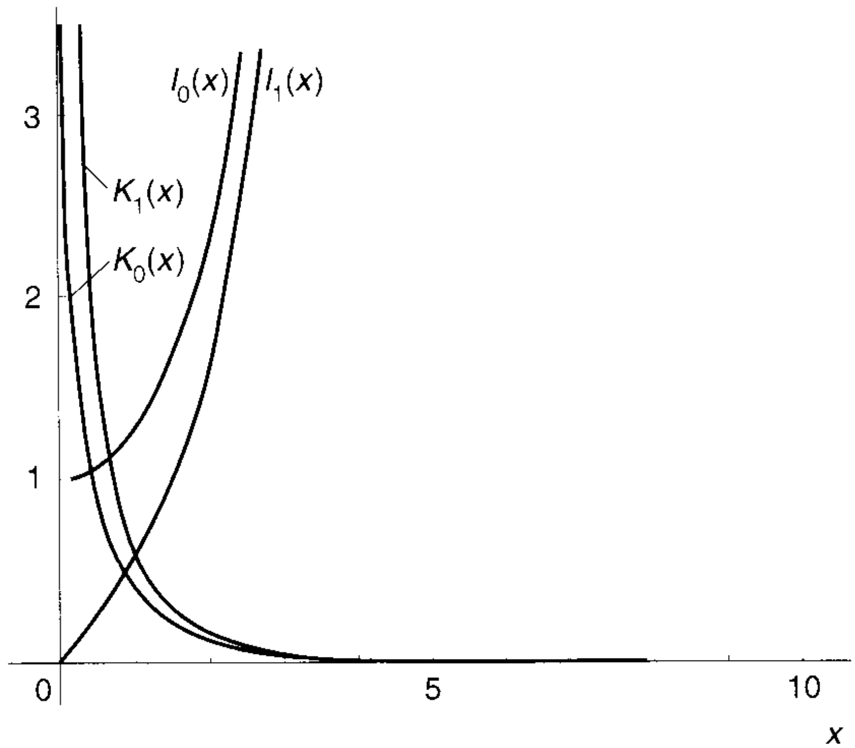


Figure 11.15 Modified Bessel functions of first $I_\nu(x)$ and the second $K_\nu(x)$ kinds.
From Izuka

Characteristic Equation for an Optical Fiber

We insist on continuity of the tangential field components E_z, E_ϕ, H_z, and H_ϕ and find:

$$\boxed{\begin{aligned}&\frac{\beta^2\nu^2}{a^2}\left(\frac{1}{\kappa^2}+\frac{1}{\gamma^2}\right)^2\\&=\left(\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right)\left(\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)}+\frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right)\end{aligned}}$$

This characteristic equation can be used with:

$$\boxed{V^2 = (\kappa a)^2 + (\gamma a)^2}, \text{ where } V = k_0 a\sqrt{n_{core}^2 - n_{clad}^2}$$

to find values for κ, γ, β, and n_{eff}.

Meridional Modes (ν=0):

For modes that correspond to bouncing meridional rays, there is no ϕ dependence. Modes are of two types – $TE_{0\mu}$ and $TM_{0\mu}$ with μ=1,2, … .

$$\boxed{\underbrace{\left(\frac{J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right)}_{\substack{\text{If we set this term = 0,} \\ E_z=E_r=0 \text{ and this is a TE mode}}}\underbrace{\left(\frac{k_0^2 n_{core}^2 J_\nu'(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_{clad}^2 K_\nu'(\gamma a)}{\gamma K_\nu(\gamma a)}\right)}_{\substack{\text{If we set this term = 0,} \\ H_z=H_r=0 \text{ and this is a TM mode}}} = 0}$$

Skew Modes ($\nu\neq0$):

These modes have radial structure. The modes have both $E_z\neq0$ and $H_z\neq0$ and thus are called “hybrid” modes. The hybrid modes are of two types labeled $EH_{\nu\mu}$ and $HE_{\nu\mu}$, depending on the whether E_z or H_z is dominant, respectively.

Field Distributions in Optical Fibers

Let's examine the mode profiles in the plane z=0:

TE Modes:	TM Modes:
$E_r \propto 0$	$E_r \propto -J_1\left(\kappa_{0\mu} r\right)$
$E_\phi \propto J_1\left(\kappa_{0\mu} r\right)$	$E_\phi \propto 0$
$H_r \propto -J_1\left(\kappa_{0\mu} r\right)$	$H_r \propto 0$
$H_\phi \propto 0$	$H_\phi \propto -J_1\left(\kappa_{0\mu} r\right)$

There is no azimuthal variation for either type of mode.

Example, TM_{01} Mode:

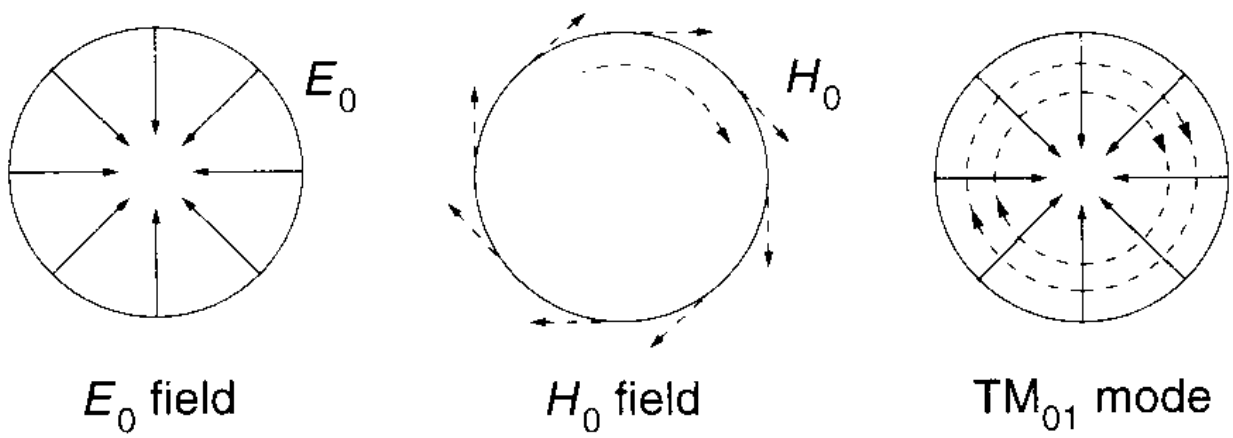


Figure 11.21. All figures (unless noted) and the table in this lecture are from Elements of Photonics, Volume II.

$J_1(\kappa_{01} r)$ has a zero at the origin and one maximum in the core.

$EH_{\nu\mu}$ Modes:

$$\boxed{\begin{aligned}
E_r &\propto -J_{\nu+1}\left(\kappa_{\nu\mu} r\right)\cos\nu\phi \\
E_\phi &\propto -J_{\nu+1}\left(\kappa_{\nu\mu} r\right)\sin\nu\phi \\
H_r &\propto J_{\nu+1}\left(\kappa_{\nu\mu} r\right)\sin\nu\phi \\
H_\phi &\propto -J_{\nu+1}\left(\kappa_{\nu\mu} r\right)\cos\nu\phi
\end{aligned}}$$

$HE_{\nu\mu}$ Modes:

$$\boxed{\begin{aligned}
E_r &\propto J_{\nu-1}\left(\kappa_{\nu\mu} r\right)\cos\nu\phi \\
E_\phi &\propto -J_{\nu-1}\left(\kappa_{\nu\mu} r\right)\sin\nu\phi \\
H_r &\propto J_{\nu-1}\left(\kappa_{\nu\mu} r\right)\sin\nu\phi \\
H_\phi &\propto J_{\nu-1}\left(\kappa_{\nu\mu} r\right)\cos\nu\phi
\end{aligned}}$$

Example - the HE_{21} mode:

$$E_r \propto J_1(\kappa_{21} r)\cos 2\phi$$
$$E_\phi \propto -J_1(\kappa_{21} r)\sin 2\phi$$
$$H_r \propto J_1(\kappa_{21} r)\sin 2\phi$$
$$H_\phi \propto J_1(\kappa_{21} r)\cos 2\phi$$

E is purely radial for $\phi = 0, \pi/2, \pi$, and $3\pi/2$.
E is purely azimuthal for $\phi = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$.
H looks like E rotated counter clockwise by $\pi/4$.
$J_1(K_{21}r)$ has a zero at the origin and one maximum in the core.

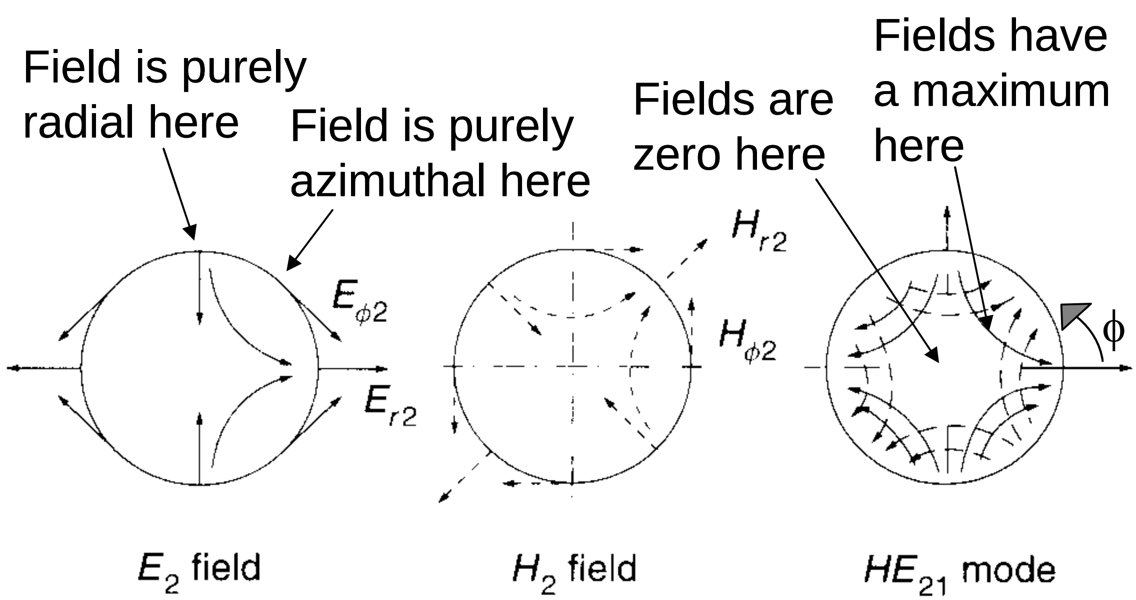


Figure 11.21.

Linearly Polarized (LP) Optical Fiber Modes

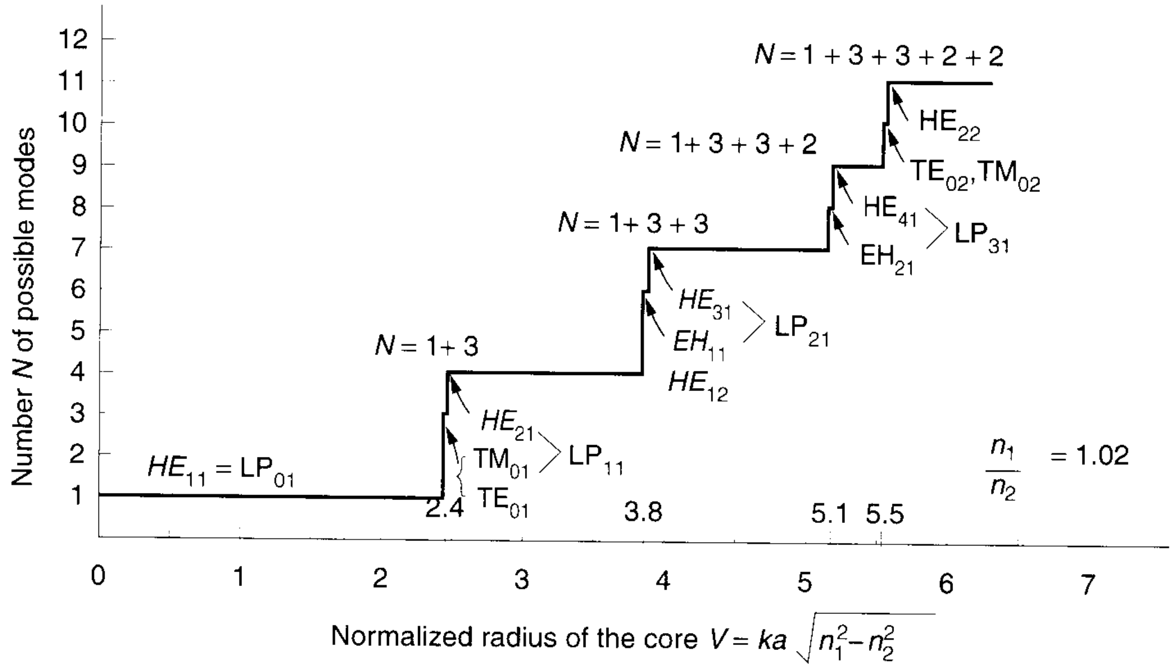


Figure 11.18 Number of possible modes in the step-index fiber as a function of the normalized radius of the core.

It is customary in the theory of optical fibers to make the "weakly guiding approximation" $n_1 = n_2$ (the refractive index of the core equal the refractive index of the cladding) because:

1. It simplifies the characteristic equation for the modes.

2. It leads to the concept of linearly polarized modes.

In the weakly guiding approximation the large steps in Figure 11.18 become not jagged as modes become degenerate (i.e. they have the same propagation constant). The degenerate modes can be added together to form new modes.

Can we construct a set of linearly polarized modes?

→ Yes. This is good because polarized light from a laser would excite these superpositions of true fiber modes.

HE_{11} is already linearly polarized.

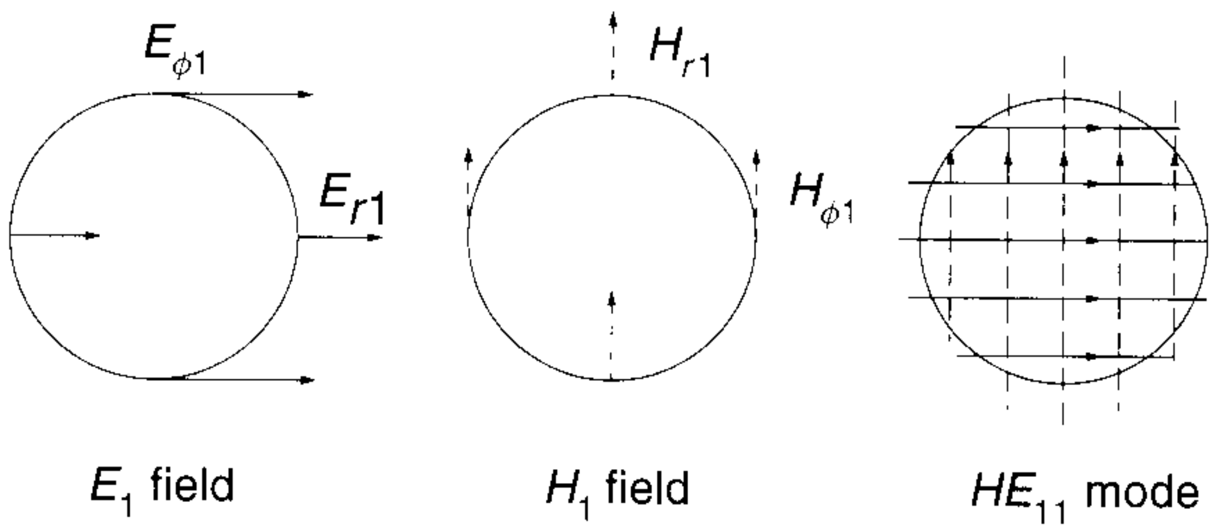


Figure 11.21 in Elements of Photonics, Volume II.

Other LP modes can be constructed from sums of the EH and HE modes that have the same propagation constant.

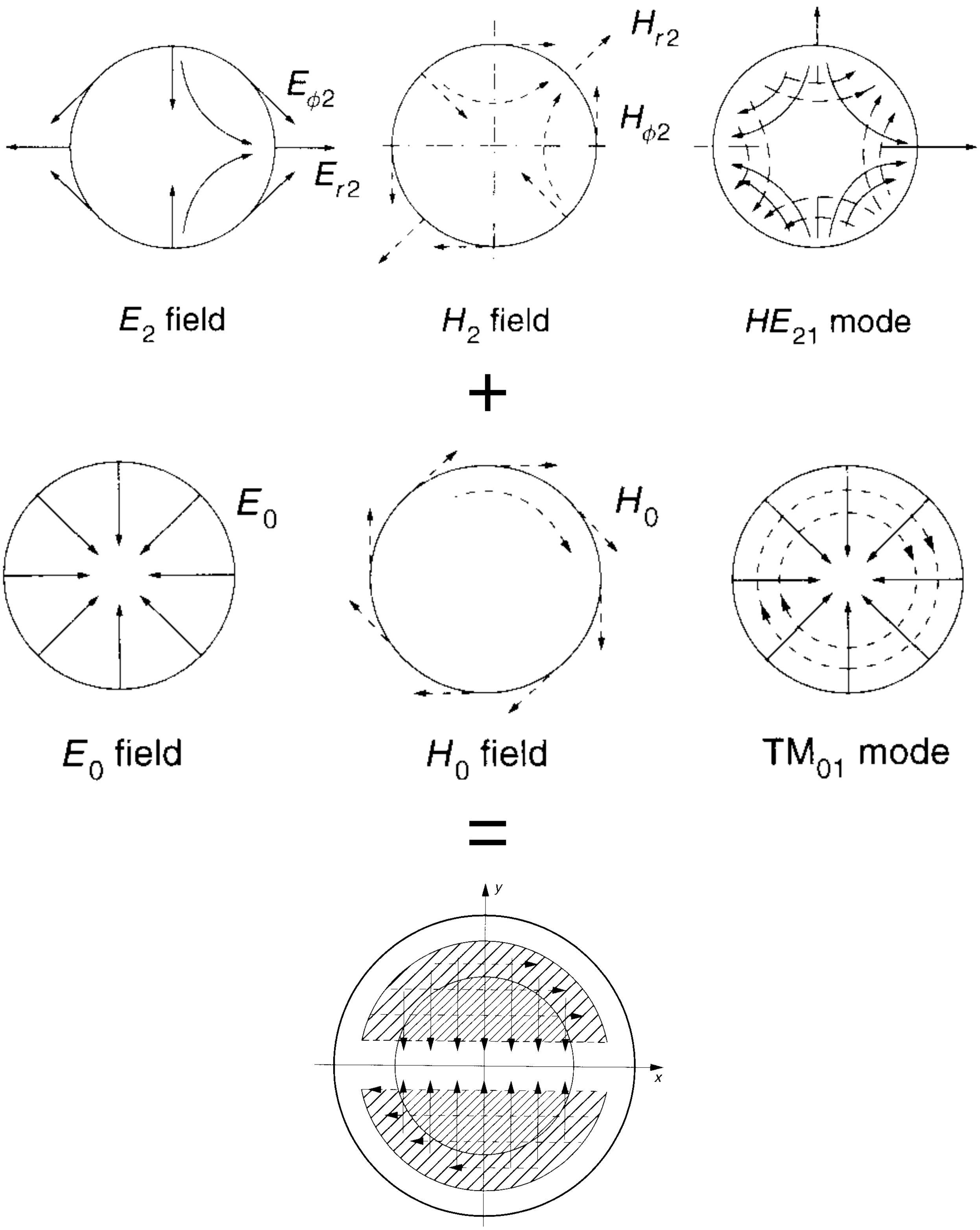


Figure 11.22 Field pattern of the LP_{11} mode.

$(m\mu)$	(01)	(11)	(21)	(02)
$\mathrm{LP}_{m\mu}$ designation Hybrid mode designation	LP_{01} HE_{11}	LP_{11} HE_{21} TE_{01} TM_{01}	LP_{21} HE_{31} EH_{11}	LP_{02} HE_{12}

Construction and Labeling Rules:

$\mathrm{LP}_{0\mu} = \mathrm{HE}_{1\mu}$

$\mathrm{LP}_{1\mu} = \mathrm{HE}_{2\mu} + \mathrm{TE}_{0\mu}$ or $\mathrm{HE}_{2\mu} + \mathrm{TM}_{0\mu}$

$\mathrm{LP}_{m\mu} = \mathrm{HE}_{m+1,\mu} + \mathrm{EH}_{m-1,\mu}$

Fiber Mode Degeneracy and Number of Modes

Degeneracy of the Hybrid Modes

Table 5.1 Modes in a circular dielectric waveguide

modes	cutoff condition	$T_c a$	number of modes	total number of modes
HE_{11}	$Ta = 0$	0	2	2
TE_{01},TM_{01},HE_{21}	$J_0(Ta)_1 = 0$	2.405	4	6
HE_{12},EH_{11},HE_{31}	$J_1(Ta)_1 = 0$	3.832	6	12
EH_{21},HE_{41}	$J_2(Ta)_1 = 0$	5.136	4	16
TE_{02},TM_{02},HE_{22}	$J_0(Ta)_2 = 0$	5.520	4	20
EH_{31},HE_{51}	$J_3(Ta)_1 = 0$	6.38	4	24
HE_{13},EH_{12},HE_{32}	$J_1(Ta)_2 = 0$	7.01	6	30
EH_{41},HE_{61}	$J_4(Ta)_1 = 0$	7.58	4	34
EH_{22},HE_{42}	$J_2(Ta)_2 = 0$	8.41	4	38
TE_{03},TM_{03},HE_{23}	$J_0(Ta)_3 = 0$	8.65	4	42
EH_{51},HE_{71}	$J_5(Ta)_1 = 0$	8.71	4	46
EH_{32},HE_{52}	$J_3(Ta)_2 = 0$	9.76	4	50
EH_{61},HE_{81}	$J_6(Ta)_1 = 0$	9.93	4	54

The number of modes includes the number of polarization degeneracy

In a multi-mode fiber, there are usually dozens, even hundreds, of guided modes, but in a single-mode fiber there is only one guided mode, i.e., the HE_{11} mode.

From Electromagnetic Theory for Microwaves and Optoelectronics, Keqian Zhang and Dejie Li

The $TE_{0\mu}$ and $TM_{0\mu}$ modes are not degenerate.

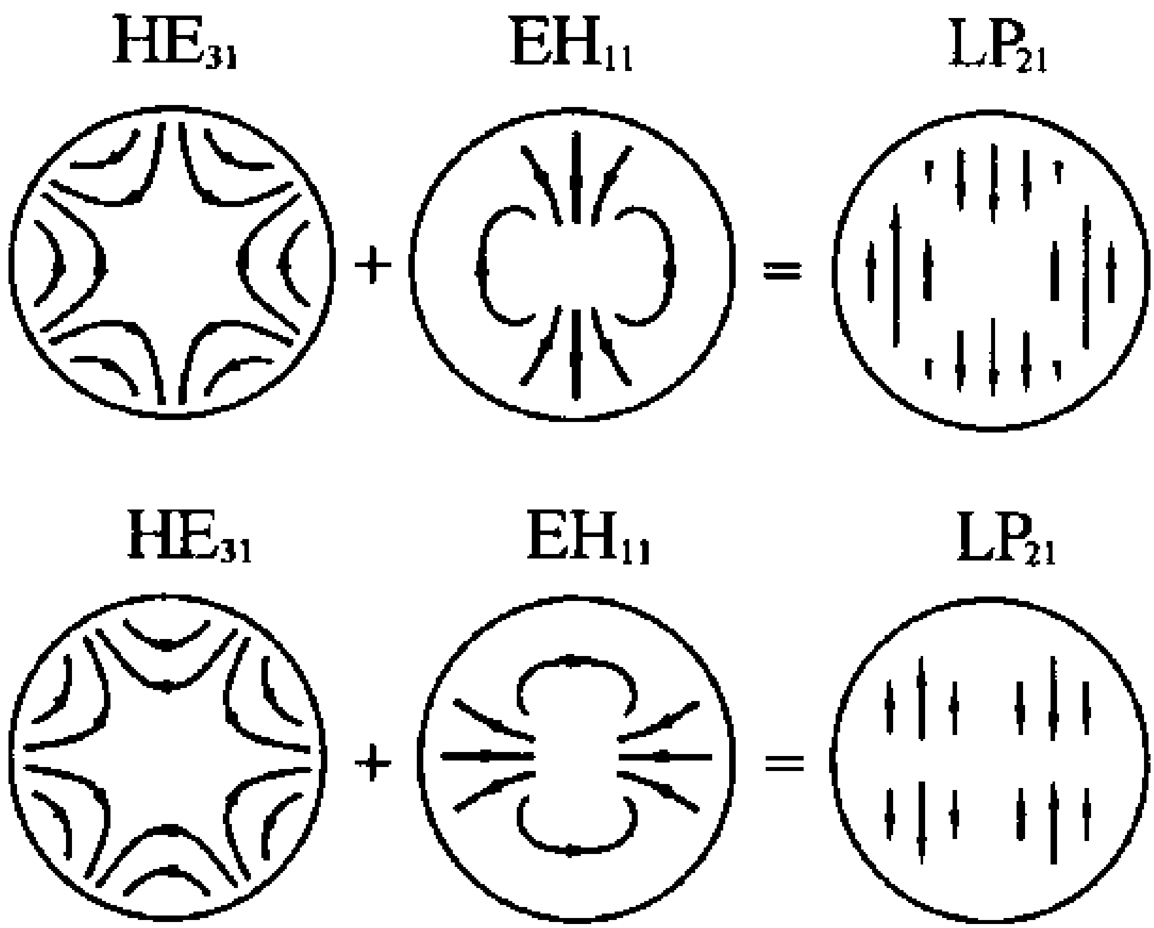
The hybrid $EH_{\nu\mu}$ and $HE_{\nu\mu}$ modes are two-fold degenerate.

Degeneracy of the LP Modes

The $LP_{0\mu}$ modes are the $HE_{1\mu}$ modes, so they are two-fold degenerate.

The $LP_{1\mu}$ modes are formed by summing $HE_{2\mu}$ + $TE_{0\mu}$ or $HE_{2\mu}$ + $TM_{0\mu}$, so they are four-fold degenerate.

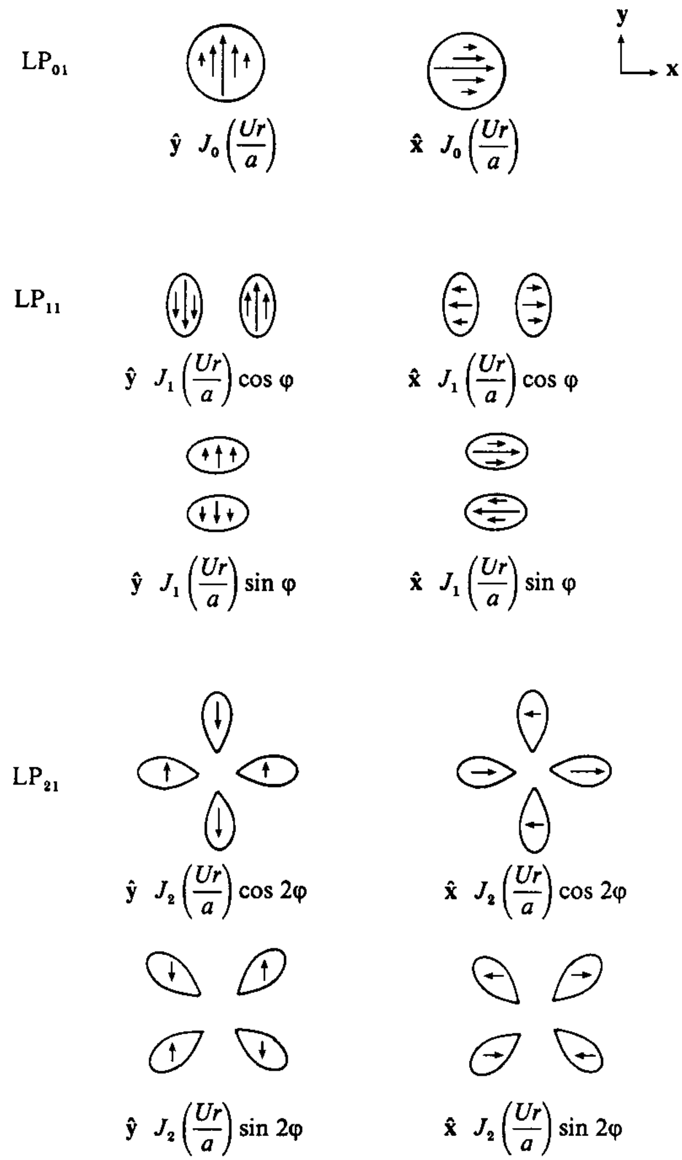
The $LP_{m\mu}$ modes with m > 1 are formed by summing $HE_{m+1,\mu}$ + $EH_{m-1,\mu}$, so they are four-fold degenerate.



Two of the 4 LP_{21} modes that can be formed from HE_{31} and EH_{11} modes.

From Electromagnetic Theory for Microwaves and Optoelectronics, Keqian Zhang and Dejie Li

Fig. 8.7: Schematic of the modal field patterns for some low-order modes in a step index fiber. The arrows represent the direction of the electric field.



From Introduction to Fiber Optics, Ghatak and Thyagarajan

Number of Modes

For large V, the number of LP or hybrid of modes is ~ $4V^2/\pi^2$.