The Wave Equation

$$\nabla \cdot \vec{D} = \rho$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Maxwell's Equations

$$\vec{B} = \mu \vec{H}$$
$$\vec{D} = \varepsilon \vec{E}$$

Constitutive Equations

Note: **E** and **B** are the macroscopic electric and magnetic fields,
D and **H** are the derived fields.

$$\nabla \times \left(\nabla \times \vec{E} \right) = \nabla \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$= -\mu \nabla \times \frac{\partial \vec{H}}{\partial t}$$

$$= -\mu \frac{\partial}{\partial t} \left(\nabla \times \vec{H} \right)$$

$$= -\mu \frac{\partial}{\partial t} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$$

$$= -\mu \frac{\partial \vec{J}}{\partial t} - \mu \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$= -\mu \frac{\partial \vec{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$= -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{(using } \vec{J} = \sigma \vec{E}\text{)}$$

$$\nabla \times \left(\nabla \times \vec{E} \right) = \underbrace{\nabla \left(\nabla \cdot \vec{E} \right)}_{\text{Usually taken to be zero}} - \nabla^2 \vec{E}$$

$\nabla \cdot \vec{E}$ is usually taken to be zero because

1. **E** varies little along the the direction of **E** for waves that are nearly plane waves.

or

2. $\nabla \cdot \vec{D} = 0$ (in a source free region)

$$= \nabla \cdot \varepsilon \vec{E} = \nabla \varepsilon \cdot \vec{E} + \varepsilon \nabla \cdot \vec{E}$$

$\Rightarrow \nabla \cdot \vec{E} = -\dfrac{\nabla \varepsilon}{\varepsilon} \cdot \vec{E} \approx 0$, when variation in ε is small

Thus

$$-\nabla^2 \vec{E} = -\mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

which can be re-arranged to give:

$$\boxed{\nabla^2 \vec{E} - \mu\sigma \frac{\partial \vec{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$ The full wave equation.

In dielectric, non-conducting media $\sigma = 0$.

Then

$$\boxed{\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

A more frequently encountered wave equation.

Mathematical Note:

$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$ for a scalar function ϕ

$$\nabla^2 \vec{E} = \nabla^2 E_x \hat{x} + \nabla^2 E_y \hat{y} + \nabla^2 E_x \hat{z}$$

Monochromatic Waves

Waves in photonics are usually monochromatic, with a frequency that stays the same across material boundaries,

i.e. $\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{j\omega t}$

Plugging this into the wave equation $\nabla^2\vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$ gives

$$
\begin{aligned}
&\nabla^2\vec{E}(\vec{r},t) - \mu\varepsilon \frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2} \\
&= \nabla^2\vec{E}(\vec{r})e^{j\omega t} - \mu\varepsilon \frac{\partial^2 \vec{E}(\vec{r})e^{j\omega t}}{\partial t^2} \\
&= \nabla^2\vec{E}(\vec{r})e^{j\omega t} + \mu\varepsilon\omega^2\vec{E}(\vec{r})e^{j\omega t} \\
&= 0
\end{aligned}
$$

$$\Rightarrow \nabla^2\vec{E}(\vec{r}) + \mu\varepsilon\omega^2\vec{E}(\vec{r}) = 0$$

or $\boxed{\nabla^2\vec{E}(\vec{r}) + n^2k_0^2\vec{E}(\vec{r}) = 0}$ Helmholtz Equation

We also have

$$\boxed{\nabla^2\vec{H}(\vec{r}) + n^2k_0^2\vec{H}(\vec{r}) = 0}$$

Note that using Cartesian coordinates we can write a scalar Helmholtz equation for each component of the electric and magnetic fields.

Bouncing Plane Wave Picture for Guided Waves

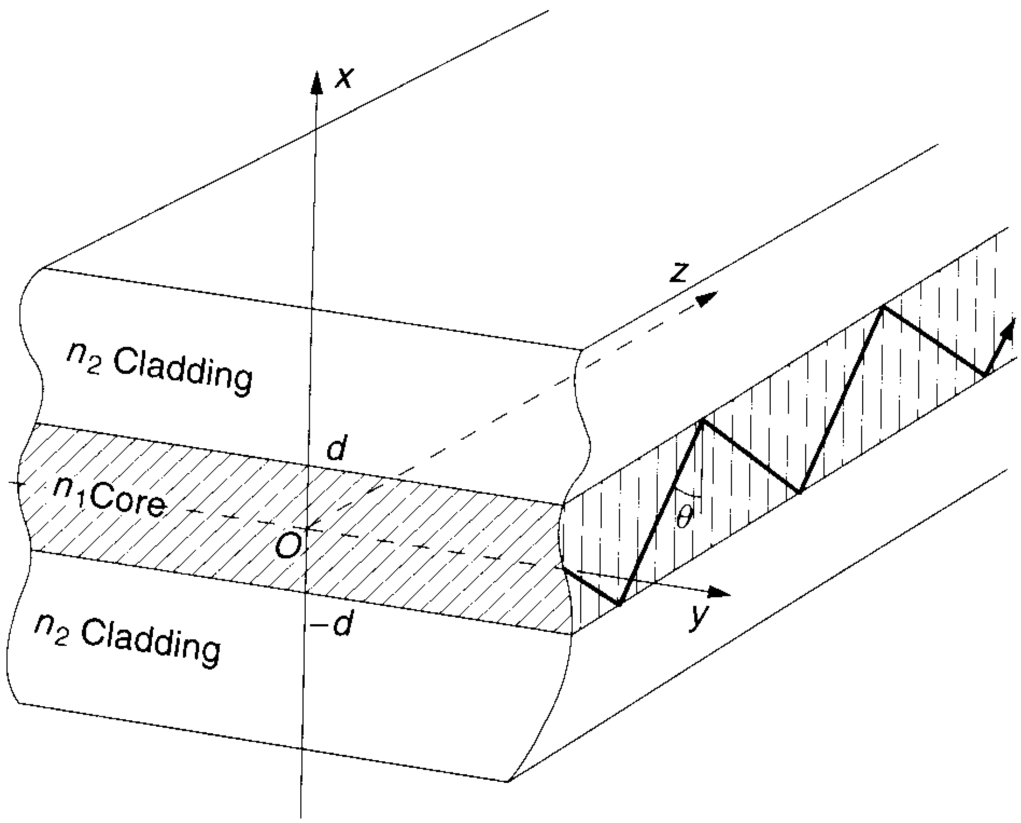

Figure 9.1 Geometry of the slab optical guide.

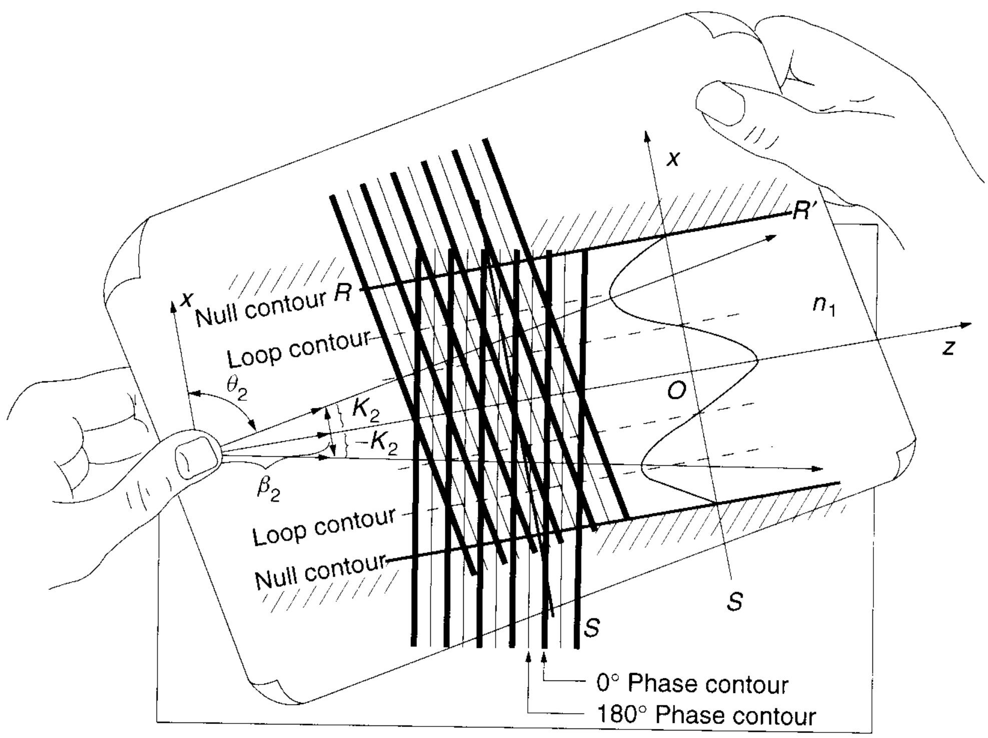

Figure 9.4 Composition of the TM_2 mode in terms of component plane waves. The effective index of refraction is $N = n_1 \sin\theta_2$.

There is a discrete set of spatial modes.

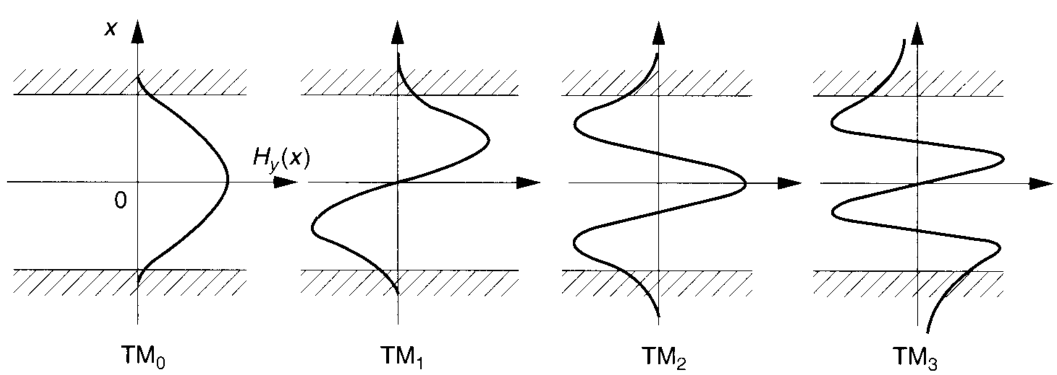

Figure 9.3 Distribution of the H_y field in the slab optical guide. The field distributions correspond to the modes in Fig. 9.2.

Why do we get a discrete set of spatial modes?

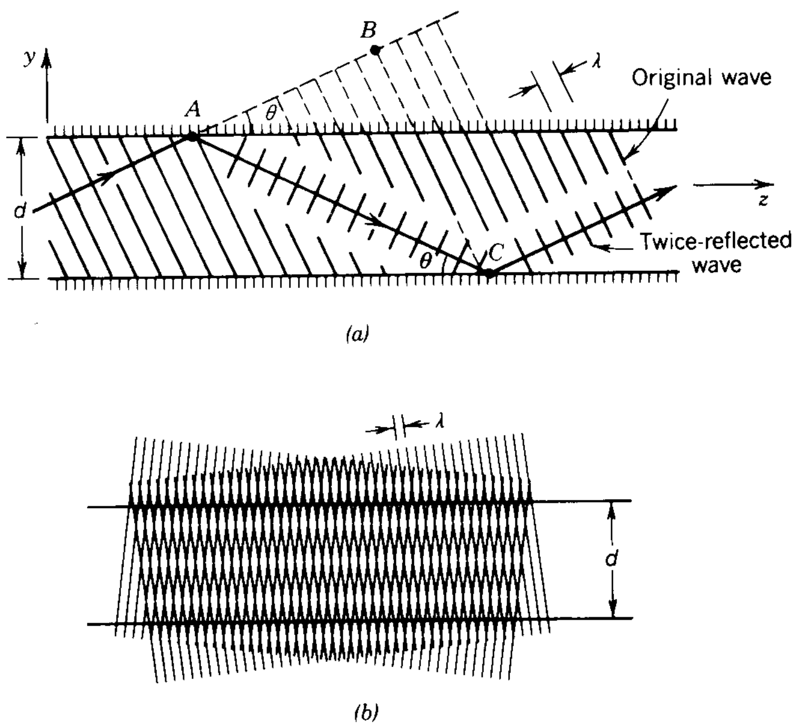

Figure 7.1-2 (*a*) Condition of self-consistency: as a wave reflects twice it duplicates itself. (*b*) At angles for which self-consistency is satisfied, the two waves interfere and create a pattern that does not change with z.

(Figure is from Fundamentals of Photonics by Saleh and Teich)

Wave Optics Approach to the Slab Waveguide

We look for solutions of the form

$$\vec{E} = \vec{E}_0(x)e^{j(\beta z - \omega t)}$$

$$\vec{H} = \vec{H}_0(x)e^{j(\beta z - \omega t)}$$

where β is called the "propagation constant".

We find solutions of two types:

1. Transverse Electric (TE) Modes

$E_z = 0$

$E_y = E_Y(x)$, $E_x = 0$

$$H_x = -\frac{\beta}{\omega\mu_0}E_y, \;\; H_z = \frac{j}{\omega\mu_0}\frac{dE_y}{dx}, \;\; H_y = 0$$

2. Transverse Magnetic (TM) Modes

$H_z = 0$,

$H_y = H_Y(x)$, $H_x = 0$

$$E_x = \frac{\beta}{\omega\varepsilon_0 n^2}H_y, \;\; E_z = -\frac{j}{\omega\varepsilon_0 n^2}\frac{dH_y}{dx}, \;\; E_y = 0$$

TM Modes for the Slab Waveguide

$$\vec{H} = H_y(x)e^{j(\beta z-\omega t)}\hat{y}$$

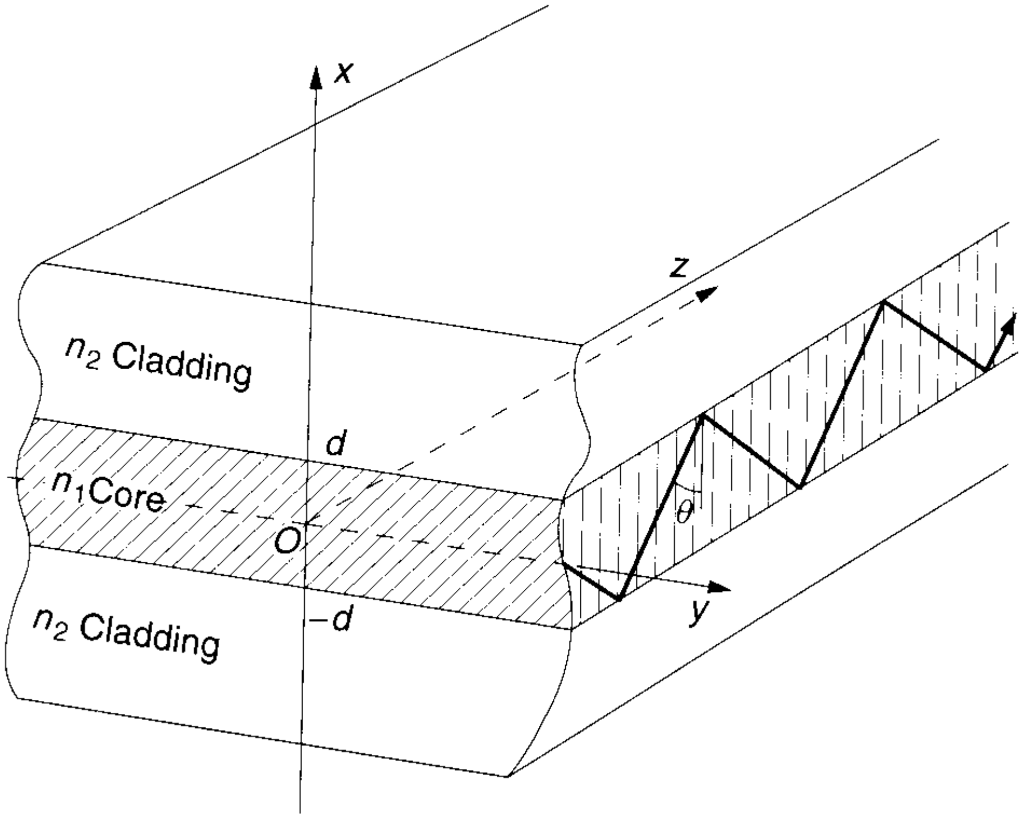

Figure 9.1 Geometry of the slab optical guide.

$$\nabla^2\vec{H}\left(\vec{r}\right)+n^2k_0{}^2\vec{H}\left(\vec{r}\right)=0 \text{ Helmholtz Equation}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)H_y(x)e^{j\beta z}\hat{y}+n^2k^2H_y(x)e^{j\beta z}\hat{y}$$

$$=\frac{\partial^2}{\partial x^2}H_y(x)e^{j\beta z}\hat{y}+H_y(x)\left(-\beta^2e^{j\beta z}\right)\hat{y}+n^2k^2H_y(x)e^{j\beta z}\hat{y}$$

$$=0$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2} H_y(x) + \left(n^2 k_0{}^2 - \beta^2\right) H_y(x) = 0}$$

For confined modes we must have:

$n_1{}^2 k_0{}^2 - \beta^2 = \kappa^2$ for $|x| < d$ (in the core)

and solutions are of the form:

$H_y(x) = A\cos(\kappa x)$ ("Even" or "Symmetric" Modes)
or
$H_y(x) = B\sin(\kappa x)$ ("Odd" or "Anti-Symmetric" Modes)

We also must have:

$n_2{}^2 k_0{}^2 - \beta^2 = -\gamma^2$ for $|x| > d$ (in the cladding)

so that solutions are of the form:

$H_y(x) = Ce^{-\gamma x}$ for $x > d$
and
$H_y(x) = De^{\gamma x}$ for $x < -d$

We use the boundary conditions

Tangential **H** and **E** must be continuous across the dielectric interfaces.

to solve for C and D in terms of A or B and to derive characteristic equations.

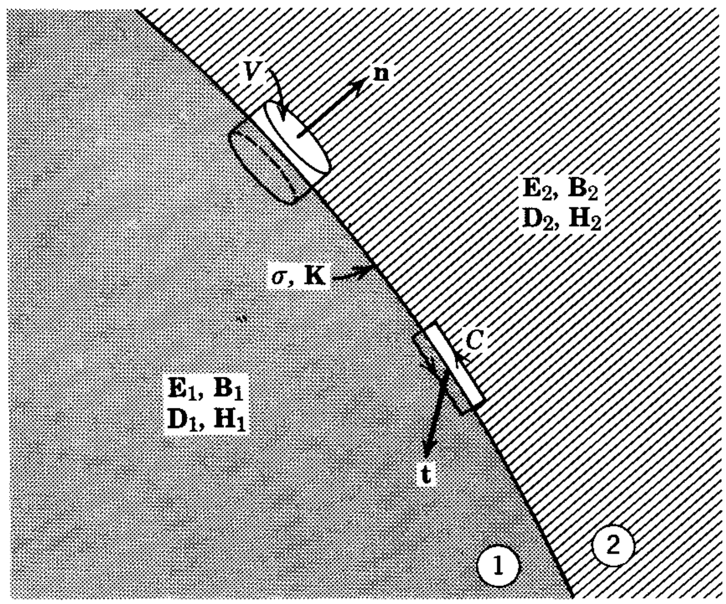

Fig. I.4 Schematic diagram of boundary surface (heavy line) between different media. The boundary region is assumed to carry idealized surface charge and current densities σ and **K**. The volume V is a small pillbox, half in one medium and half in the other, with the normal **n** to its top pointing from medium 1 into medium 2. The rectangular contour C is partly in one medium and partly in the other and is oriented with its plane perpendicular to the surface so that its normal **t** is tangent to the surface.

From Classical Electrodynamics, 2nd Ed., J. D. Jackson

Ampere's Law and the curve C give $\boldsymbol{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0$

Faraday's Law and the volume V give
$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{K}$

Continuity of H_y gives for the Symmetric Modes:

$$A\cos(\kappa d) = Ce^{-\gamma d} \text{ and } C = \frac{A\cos(\kappa d)}{e^{-\gamma d}}$$

(the symmetry of the mode about x = 0 gives D = C)

Continuity of $E_z = -\frac{j}{\omega\varepsilon_0 n^2}\frac{dH_y}{dx}$ gives:

$$\frac{1}{n_1^2}\kappa A\sin(\kappa d) = \frac{1}{n_2^2}\gamma Ce^{-\gamma d}$$

Dividing the 2nd equation by the first gives:

$$\boxed{\left(\frac{n_2}{n_1}\right)^2 \kappa d\tan(\kappa d) = \gamma d \text{ for Symmetric Modes}}$$

In a similar way we can derive:

$$C = \frac{B\sin(\kappa d)}{e^{-\gamma d}} \text{ with } D = -C$$

and

$$\boxed{-\left(\frac{n_2}{n_1}\right)^2 \kappa d\cot(\kappa d) = \gamma d \text{ for Anti-Symmetric Modes}}$$

These are called the "characteristic equations" for the TM modes. They are used with

$$(\kappa d)^2 + (\gamma d)^2 = V^2, \text{ where } V \equiv k_0 d\sqrt{n_1^2 - n_2^2},$$

to find γ, κ and $,\beta$

V is called the "normalized frequency", "V parameter", "V number", and "normalized thickness" of the waveguide.

Caution: Some authors use the full thickness of the core for d in the definition for V

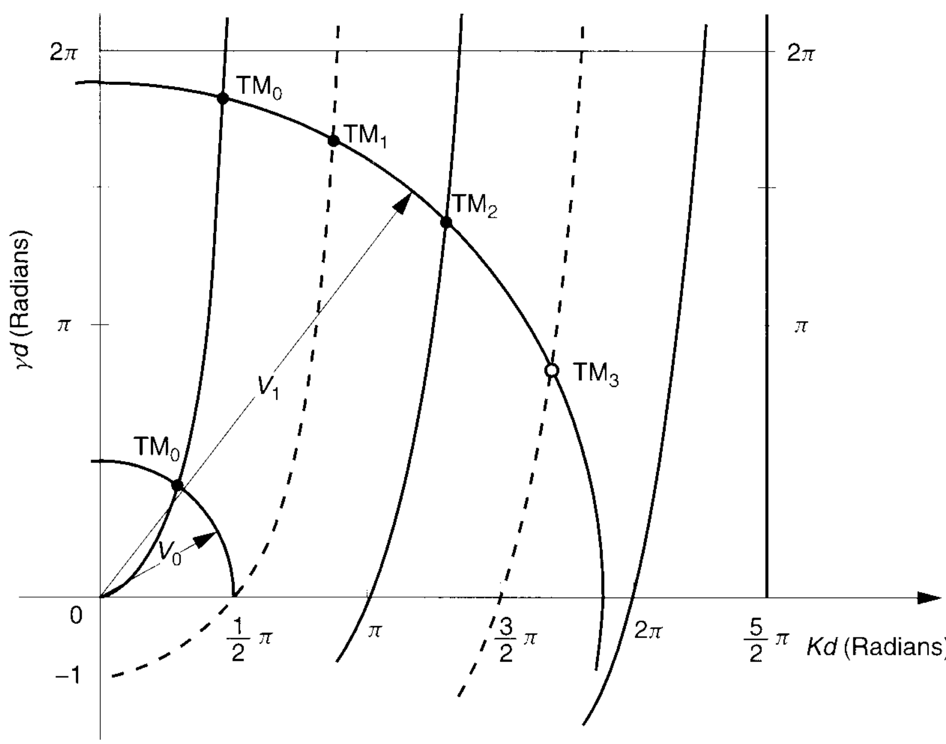

Figure 9.2 Graphical solutions for the even TM modes ($\gamma d = n^2 Kd \tan Kd$) in solid lines, and odd TM modes ($-\gamma d = n^2 Kd \cot Kd$) in dashed lines. $V = kd\sqrt{n_1^2 - n_2^2}$.

From Iizuka, Elements of Photonics Vol. II

Notice that there is one confined mode if $V \leq \pi/2$, two confined modes if $\pi/2 < V \leq \pi$, etc. This is one of the reasons for defining and calculating V $\rightarrow$ it gives you the number of confined modes.