Coupled Waveguides – Super Modes

\[ \beta_o = 10.787 \text{ } \mu\text{m}^{-1} \quad \beta_e = 10.788 \text{ } \mu\text{m}^{-1} \]

Figure 9.20  Calculation of the field \( H_y(x) \) along the \( x \) axis inside the five-layer medium. (Calculation courtesy of R. James.)

\[ L = \frac{\pi}{\beta_e - \beta_o} \]

(L = 4.41 mm for the example above)
Coupled Mode Theory – Perturbation Theory

\[ \vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \vec{P}_{\text{waveguide}} + \vec{P}_{\text{pert}} \]

or

\[ \vec{D} = \varepsilon \vec{E} + \vec{P}_{\text{pert}} \]

The wave equation becomes:

\[ \nabla^2 \vec{E} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}_{\text{pert}}}{\partial t^2} \]

We can write an arbitrary guided wave as:

\[ E_{\text{coupled}} = A(z)E_A(x)e^{-j(\beta z - \omega t)} + B(z)E_B(x)e^{-j(\beta z - \omega t)} \]

where it is implied that we must eventually take the real part of imaginary quantities.

We plug \( E_{\text{coupled}} \) into the wave equation (with perturbation) and find:

\[ \left( \frac{\partial^2 A}{\partial z^2} - 2j \beta \frac{\partial A}{\partial z} E_A(x) + \frac{\partial^2 B}{\partial z^2} - 2j \beta \frac{\partial B}{\partial z} E_B(x) \right)e^{-j(\beta z - \omega t)} \]

\[ = \mu_0 \frac{\partial^2 P_{\text{pert}}}{\partial t^2} \]

(more detail in appendix below)
A common approximation is the “Slowly Varying Envelope Approximation” - \[
\frac{\partial^2 A}{\partial z^2} \quad \text{and} \quad \frac{\partial^2 B}{\partial z^2} \quad \text{and} \quad \beta \frac{\partial A}{\partial z} \quad \text{and} \quad \beta \frac{\partial B}{\partial z}
\] which gives:

\[
\left(-2j \beta \frac{\partial A}{\partial z} E_A(x) - 2j \beta \frac{\partial B}{\partial z} E_B(x)\right) e^{-j(\beta z - \omega t)} = \mu_0 \frac{\partial^2 P_{\text{pert}}}{\partial t^2}
\]

Multiplying on the right by \(E_A(x)\) or \(E_B(x)\) and integrating over \(x\) gives the starting equations for coupled mode calculations:

\[
\frac{\partial A}{\partial z} e^{-j(\beta z - \omega t)} = \frac{j}{2\beta} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} P_{\text{pert}}(x,t) E_A(x) \, dx
\]

and

\[
\frac{\partial B}{\partial z} e^{-j(\beta z - \omega t)} = \frac{j}{2\beta} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} P_{\text{pert}}(x,t) E_B(x) \, dx
\]

where we used the fact that the mode profiles are orthogonal and assumed they are normalized:

\[
\int_{-\infty}^{\infty} E_A(x) E_B(x) \, dx = 0
\]

and

\[
\int_{-\infty}^{\infty} E_A(x) E_A(x) \, dx = \int_{-\infty}^{\infty} E_B(x) E_B(x) \, dx = 1
\]
Coupling Between Degenerate Modes in Parallel Slab Waveguides:

![Diagram of coupled waveguides](image)

**Figure 10.9** A directional coupler can be made by placing two waveguides in close proximity to one another for a finite distance. The evanescent field from one waveguide overlaps the core of the second waveguide, leading to coupling.

From *Fundamentals of Optoelectronics*, Pollock

The "perturbation" seen by the mode A is due to mode B in the core of B.

\[
P_{\text{pert}} = \varepsilon_0 \left( n_1^2 - n_2^2 \right) B(z) E_B(x) e^{-j(\beta z - \omega t)}
\]

Plugging this into the first starting equation for coupled mode calculations gives:

\[
\frac{\partial A}{\partial z} = -j\kappa B
\]

where the coupling constant is:
\[ \kappa = \frac{\omega^2 \varepsilon_0}{2 \beta} \left( n_1^2 - n_2^2 \right) \int_{\text{core of } B} E_B(x) E_A(x) \, dx \]

similarly

\[ \frac{\partial B}{\partial z} = -j \kappa A \]

The equations show that the optical power oscillates between the two slab guides, transferring completely in a length \( \pi/2\kappa \).

![Graph showing the power oscillation](image)

**Figure 10.11** The power in waveguide \( B \) varies sinusoidally as a function of coupling length \( a \).

From *Fundamentals of Optoelectronics*
Connection with Supermode Approach

\[ \beta_e - \beta_o = 2\kappa \]

Note: Coupled mode theory can be used to model many other coupled wave phenomena including wave propagation in distributed feedback lasers.

**FIGURE 2.1.** Schematic structure of a GaAlAs/GaAs distributed feedback laser achieving cw operation at room temperature (after Ref. 6 © 1976 IEEE).

*Figure from Semiconductor Lasers: Past, Present, and Future, G. P. Agrawal ed.*
Appendix

We start with the driven vector wave equation

$$\nabla^2 \vec{E} - \mu_0 \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}_{\text{pert}}}{\partial t^2}$$

and a coupled guided wave

$$E_{\text{coupled}} = A(z)E_A(x)e^{-j(\beta z - \omega t)} + B(z)E_B(x)e^{-j(\beta z - \omega t)},$$

a scalar quantity that represents the either a TE or TM mode.

Noting that the ortho-normal guided modes $E_A(x)$ and $E_A(x)$ satisfy the homogeneous wave equation

$$\nabla^2 E_{A,B}(x)e^{-j(\beta z - \omega t)} - \mu_0 \varepsilon \frac{\partial^2 E_{A,B}(x)e^{-j(\beta z - \omega t)}}{\partial t^2} = 0.$$  

We plug $E(x)$ into the wave equation with perturbation and get

$$\left(\frac{\partial^2 A}{\partial z^2} - 2j \beta \frac{\partial A}{\partial z} E_A(x) + \frac{\partial^2 B}{\partial z^2} - 2j \beta \frac{\partial B}{\partial z} E_B(x)\right)e^{-j(\beta z - \omega t)}$$

$$= \mu_0 \frac{\partial^2 P_{\text{pert}}}{\partial t^2}$$
\[ \nabla^2 E_A(x) e^{-j(\beta z - \omega t)} \] 
\[ - \mu_0 \varepsilon \frac{\partial^2 E_A(x) e^{-j(\beta z - \omega t)}}{\partial t^2} \] 
\[ = 0 \]
\[ + \nabla^2 E_B(x) e^{-j(\beta z - \omega t)} \] 
\[ - \mu_0 \varepsilon \frac{\partial^2 E_B(x) e^{-j(\beta z - \omega t)}}{\partial t^2} \] 
\[ = 0 \]
\[ + \left[ \frac{\partial^2 A}{\partial z^2} - 2j \beta \frac{\partial A}{\partial z} \right] E_A(x) e^{-j(\beta z - \omega t)} \]
\[ + \left[ \frac{\partial^2 B}{\partial z^2} - 2j \beta \frac{\partial B}{\partial z} \right] E_B(x) e^{-j(\beta z - \omega t)} \]
\[ = \mu_0 \frac{\partial^2 P_{pert}}{\partial t^2} \]

Then the “Slowly Varying Envelope Approximation” gives:
\[ \left( -2j \beta \frac{\partial A}{\partial z} E_A(x) - 2j \beta \frac{\partial B}{\partial z} E_B(x) \right) e^{-j(\beta z - \omega t)} = \mu_0 \frac{\partial^2 P_{pert}}{\partial t^2} \]