Theory of Optical Modes in Step Index Fibers

Figure 11.13 Optical fiber in a cylindrical coordinate system.
$$n = \begin{cases} 
  n_{\text{core}} & \text{inside the core} \\
  n_{\text{clad}} & \text{inside the cladding}
\end{cases}$$

$$\vec{E} = E_r \hat{r} + E_\phi \hat{\phi} + E_z \hat{z}$$

$$\vec{H} = H_r \hat{r} + H_\phi \hat{\phi} + H_z \hat{z}$$

We find the modes by looking for solutions of:

$$\nabla^2 \vec{E} + (nk_0)^2 \vec{E} = \left( \nabla^2 E_r - \frac{2}{r^2} \frac{\partial E_\phi}{\partial \phi} - \frac{E_r}{r^2} + (nk_0)^2 E_r \right) \hat{r}$$

$$+ \left( \nabla^2 E_\phi + \frac{2}{r^2} \frac{\partial E_r}{\partial \phi} - \frac{E_\phi}{r^2} + (nk_0)^2 E_\phi \right) \hat{\phi}$$

$$+ \left( \nabla^2 E_z + (nk_0)^2 E_z \right) \hat{z} = 0$$

The equations have a simple physical interpretation.

**Figure 5.2** A radial field at one point in the waveguide will become an azimuthal field at another location. Notice that the field is not converted between the components by reflection, but by propagation through the coordinate system. (from Pollock)
Since the equations for $E_r$ and $E_\phi$ are coupled, we first solve for $E_z$. $H_z$ is a solution of the same Helmholtz equation and its solutions have the same form. We find all other field components from $E_z$ and $H_z$ using Maxwell’s equations.

We look for solutions of the form:

$$E_z = R(r) \Phi(\phi) Z(z)$$
In the core we find:

\[ Z(z) = ae^{-j\beta z} + be^{j\beta z} \]
\[ \Phi(\phi) = ce^{j\nu\phi} + de^{-j\nu\phi} \]
\[ R(r) = eJ_\nu(\kappa r) + fN_\nu(\kappa r) \]

where \( \kappa^2 = \left( n_{\text{core}} k_0 \right)^2 - \beta^2 \), and \( \nu = 0,1,2... \)

We can simplify these noting that:

- Often we have only forward going waves (b=0)
- The \( N_\nu(\kappa r) \) solution goes to minus infinity at \( r = 0 \) so it is unphysical (f=0)

![Figure 11.14](image)

**Figure 11.14** Zero- and first-order Bessel functions of the second kind.

- We need both the \( e^{j\nu\phi} \) and \( e^{-j\nu\phi} \) terms to describe the \( \phi \) dependence of the eigenmodes, but we can limit the discussion to the \( e^{j\nu\phi} \) solution with the understanding
that a mode with $e^{j\nu\phi}$ dependence can be found from the $e^{jv\phi}$ mode by rotating the fiber.

Then we can write:

$$E_z = AJ_{v}(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$
$$H_z = BJ_{v}(\kappa r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

In the cladding region we find:

$$E_z = CK_{v}(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$
$$H_z = DK_{v}(\gamma r)e^{j\nu\phi}e^{-j\beta z} + c.c.$$

where $-\gamma^2 = (n_{clad}k_0)^2 - \beta^2$

*Figure 4.4.* The first three Bessel functions of the first kind, $J_0(\kappa r)$, and of the second kind, $K_0(\gamma r)$.

*From Pollock and Lipson*
Characteristic Equation for an Optical Fiber

We insist on continuity of the tangential field components $E_z$, $E_\phi$, $H_z$, and $H_\phi$ and find:

$$\frac{\beta^2 \nu^2}{a^2} \left( \frac{1}{\kappa^2} + \frac{1}{\gamma^2} \right)^2$$

$$= \left( \frac{J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right) \left( \frac{k_0^2 n_{\text{core}}^2 J'_\nu(\kappa a)}{\kappa J_\nu(\kappa a)} + \frac{k_0^2 n_{\text{clad}}^2 K'_\nu(\gamma a)}{\gamma K_\nu(\gamma a)} \right)$$
This characteristic equation can be used with:

\[ V^2 = (\kappa a)^2 + (\gamma a)^2, \quad \text{where} \quad V = k_0 a \sqrt{n_{\text{core}}^2 - n_{\text{clad}}^2} \]

to find values for \( \kappa, \gamma, \beta, \) and \( n_{\text{eff}}. \)

Meridional Modes (\( \nu = 0 \)):

For modes that correspond to bouncing meridional rays, there is no \( \phi \) dependence. Modes are of two types – TE\(_{0\mu} \) and TM\(_{0\mu} \) with \( \mu = 1, 2, \ldots \).

\[
\left( \frac{J'_{\nu}(\kappa a)}{\kappa J_{\nu}(\kappa a)} + \frac{K'_{\nu}(\gamma a)}{\gamma K_{\nu}(\gamma a)} \right) \left( \frac{k_0^2 n_{\text{core}}^2 J'_{\nu}(\kappa a)}{\kappa J_{\nu}(\kappa a)} + \frac{k_0^2 n_{\text{clad}}^2 K'_{\nu}(\gamma a)}{\gamma K_{\nu}(\gamma a)} \right) = 0
\]

If we set this term = 0, \( E_z - E_r = 0 \) and this is a TE mode
If we set this term = 0, \( H_z - H_r = 0 \) and this is a TM mode

Skew Modes (\( \nu \neq 0 \)):

These modes have radial structure. The modes have both \( E_z \neq 0 \) and \( H_z \neq 0 \) and thus are called “hybrid” modes. The hybrid modes are of two types labeled \( \text{EH}^{\nu\mu} \) and \( \text{HE}^{\nu\mu} \), depending on the whether \( E_z \) or \( H_z \) is dominant, respectively.
Field Distributions in Optical Fibers

Let’s examine the mode profiles in the plane z=0:

**TE Modes:**

\[
\begin{align*}
E_r & \equiv 0 \\
E_\phi & \equiv J_1(\kappa_{0\mu} r) \\
H_r & \equiv -J_1(\kappa_{0\mu} r) \\
H_\phi & \equiv 0
\end{align*}
\]

**TM Modes:**

\[
\begin{align*}
E_r & \equiv -J_1(\kappa_{0\mu} r) \\
E_\phi & \equiv 0 \\
H_r & \equiv 0 \\
H_\phi & \equiv -J_1(\kappa_{0\mu} r)
\end{align*}
\]

There is no azimuthal variation for either type of mode.

Example, TM\textsubscript{01} Mode:

\[
\begin{align*}
E_0 & \text{ field} \\
H_0 & \text{ field} \\
\text{TM}_0^1 & \text{ mode}
\end{align*}
\]

\textbf{Figure 11.21.} All figures (unless noted) and the table in this lecture are from Elements of Photonics, Volume II.

\(J_1(\kappa_{01} r)\) has a zero at the origin and one maximum in the core.
**EH_{νµ} Modes:**

\[ E_r = -J_{ν+1}(κ_{νµ} r) \cos νφ \]
\[ E_φ = -J_{ν+1}(κ_{νµ} r) \sin νφ \]
\[ H_r = J_{ν+1}(κ_{νµ} r) \sin νφ \]
\[ H_φ = -J_{ν+1}(κ_{νµ} r) \cos νφ \]

**HE_{νµ} Modes:**

\[ E_r = J_{ν-1}(κ_{νµ} r) \cos νφ \]
\[ E_φ = -J_{ν-1}(κ_{νµ} r) \sin νφ \]
\[ H_r = J_{ν-1}(κ_{νµ} r) \sin νφ \]
\[ H_φ = J_{ν-1}(κ_{νµ} r) \cos νφ \]
Example - the $HE_{21}$ mode:

\[
\begin{align*}
E_r &= J_1(\kappa r) \cos 2\phi \\
E_\phi &= -J_1(\kappa r) \sin 2\phi \\
H_r &= J_1(\kappa r) \sin 2\phi \\
H_\phi &= J_1(\kappa r) \cos 2\phi
\end{align*}
\]

$E$ is purely radial for $\phi = 0, \pi/2, \pi, \text{and } 3\pi/2$.  
$E$ is purely azimuthal for $\phi = \pi/4, 3\pi/4, 5\pi/4, \text{and } 7\pi/4$.  
$H$ looks like $E$ rotated counter clockwise by $\pi/4$.  
$J_1(K_2 r)$ has a zero at the origin and one maximum in the core.

\[\text{Figure 11.21.}\]
Linearly Polarized (LP) Optical Fiber Modes

It is customary in the theory of optical fibers to make the “weakly guiding approximation” $n_1 = n_2$ (the refractive index of the core equal the refractive index of the cladding) because:

1. It simplifies the characteristic equation for the modes.

2. It leads to the concept of linearly polarized modes.

In the weakly guiding approximation the large steps in Figure 11.18 become not jagged as modes become degenerate (i.e. they have the same propagation constant). The degenerate modes can be added together to form new modes.
Can we construct a set of linearly polarized modes?

→ Yes. This is good because polarized light from a laser would excite these superpositions of true fiber modes.

$\text{HE}_{11}$ is already linearly polarized.

![Figure 11.21 in Elements of Photonics, Volume II.](image)

Other LP modes can be constructed from sums of the EH and HE modes that have the same propagation constant.
Figure 11.22  Field pattern of the LP_{11} mode.
### Construction and Labeling Rules:

LP$_{0\mu}$ = HE$_{1\mu}$  
LP$_{1\mu}$ = HE$_{2\mu}$ + TE$_{0\mu}$ or HE$_{2\mu}$ + TM$_{0\mu}$  
LP$_{m\mu}$ = HE$_{m+1,\mu}$ + EH$_{m-1,\mu}$

<table>
<thead>
<tr>
<th>$(m\mu)$</th>
<th>(01)</th>
<th>(11)</th>
<th>(21)</th>
<th>(02)</th>
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<tbody>
<tr>
<td>LP$_{m\mu}$ designation</td>
<td>LP$_{01}$</td>
<td>LP$_{11}$</td>
<td>LP$_{21}$</td>
<td>LP$_{02}$</td>
</tr>
<tr>
<td>Hybrid mode designation</td>
<td>HE$_{11}$</td>
<td>HE$_{21}$</td>
<td>HE$_{31}$</td>
<td>HE$_{12}$</td>
</tr>
<tr>
<td>TE$_{01}$</td>
<td>EH$_{11}$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


Fiber Mode Degeneracy and Number of Modes

Degeneracy of the Hybrid Modes

<table>
<thead>
<tr>
<th>modes</th>
<th>cutoff condition</th>
<th>$T_c a$</th>
<th>number of modes</th>
<th>total number of modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>HE_{11}</td>
<td>$T_1 a = 0$</td>
<td>0</td>
<td>2</td>
<td>2</td>
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<tr>
<td>TE_{01}, TM_{01}, HE_{21}</td>
<td>$J_0(T_1 a_1) = 0$</td>
<td>2.405</td>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>HE_{12}, EH_{11}, HE_{31}</td>
<td>$J_1(T_1 a_1) = 0$</td>
<td>3.832</td>
<td>6</td>
<td>12</td>
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<tr>
<td>EH_{21}, HE_{41}</td>
<td>$J_2(T_1 a_1) = 0$</td>
<td>5.136</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>TE_{02}, TM_{02}, HE_{22}</td>
<td>$J_0(T_2 a_2) = 0$</td>
<td>5.520</td>
<td>4</td>
<td>20</td>
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<tr>
<td>EH_{31}, HE_{51}</td>
<td>$J_3(T_1 a_3) = 0$</td>
<td>6.38</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>HE_{13}, EH_{12}, HE_{32}</td>
<td>$J_1(T_2 a_2) = 0$</td>
<td>7.01</td>
<td>6</td>
<td>30</td>
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<tr>
<td>EH_{41}, HE_{61}</td>
<td>$J_4(T_1 a_4) = 0$</td>
<td>7.58</td>
<td>4</td>
<td>34</td>
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<tr>
<td>EH_{22}, HE_{42}</td>
<td>$J_2(T_2 a_2) = 0$</td>
<td>8.41</td>
<td>4</td>
<td>38</td>
</tr>
<tr>
<td>TE_{03}, TM_{03}, HE_{23}</td>
<td>$J_0(T_3 a_3) = 0$</td>
<td>8.65</td>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>EH_{51}, HE_{71}</td>
<td>$J_5(T_1 a_5) = 0$</td>
<td>8.71</td>
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<td>46</td>
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<tr>
<td>EH_{32}, HE_{52}</td>
<td>$J_3(T_2 a_3) = 0$</td>
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<td>4</td>
<td>50</td>
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<tr>
<td>EH_{61}, HE_{81}</td>
<td>$J_6(T_2 a_6) = 0$</td>
<td>9.93</td>
<td>4</td>
<td>54</td>
</tr>
</tbody>
</table>

The number of modes includes the number of polarization degeneracy

In a multi-mode fiber, there are usually dozens, even hundreds, of guided modes, but in a single-mode fiber there is only one guided mode, i.e., the HE_{11} mode.

*From Electromagnetic Theory for Microwaves and Optoelectronics, Keqian Zhang and Dejie Li*

The TE_{0μ} and TM_{0μ} modes are not degenerate.

The hybrid EH_{νμ} and HE_{νμ} modes are two-fold degenerate.

Degeneracy of the LP Modes

The LP_{0μ} modes are the HE_{1μ} modes, so they are two-fold degenerate.
The LP1_μ modes are formed by summing HE2_μ + TE0_μ or HE2_μ + TM0_μ, so they are four-fold degenerate.

The LP_{m_μ} modes with m > 1 are formed by summing HE_{m+1,μ} + EH_{m-1,μ}, so they are four-fold degenerate.

Two of the 4 LP_{21} modes that can be formed from HE_{31} and EH_{11} modes.

*From Electromagnetic Theory for Microwaves and Optoelectronics, Keqian Zhang and Dejie Li*
Number of Modes

For large $V$, the number of LP or hybrid of modes is $\sim 4V^2/\pi^2$. 